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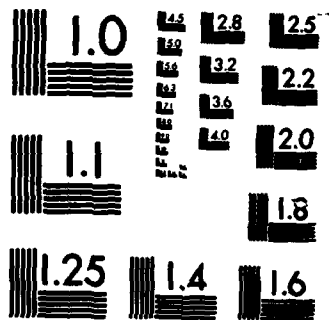
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FLUTTER TAMING -  
A NEW TOOL FOR THE  
AEROELASTIC DESIGNER

by

Luigi Morino



DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS

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BOSTON UNIVERSITY  
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by

Luigi Morino

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boundary) that flutter taming is always possible for an aeroservoelastic system that can be represented by a system of nonlinear differential equations with analytical nonlinearities. It is important to emphasize that the control system for flutter taming is fully nonlinear, and therefore it does not affect the linear behavior (in particular the stability characteristics) of the system. Hence, flutter taming can be used in conjunction with flutter suppression by active control (i.e., use of linear active control to increase the flutter speed). Applications of the theory to the case of an airfoil in supersonic flow are presented. In addition to an active control modification (use of control surface with nonlinear feedback), passive modifications (e.g., a nonlinear damper) are also investigated. *Originator-supplied Keyword includes:*

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## ABSTRACT

A new concept for the design of aeroservoelastic systems is introduced: flutter taming by nonlinear control, i.e., use of nonlinear terms in the equation to ensure that the behavior of the system beyond the flutter speed is of benign rather than destructive nature. This is accomplished by using a very simple nonlinear control law. It is shown (using a singular perturbation analysis about the stability boundary) that flutter taming is always possible for an aeroservoelastic system that can be represented by a system of nonlinear differential equations with analytical nonlinearities. It is important to emphasize that the control system for flutter taming is fully nonlinear, and therefore it does not affect the linear behavior (in particular the stability characteristics) of the system. Hence, flutter taming can be used in conjunction with flutter suppression by active control (i.e., use of linear active control to increase the flutter speed). Applications of the theory to the case of an airfoil in supersonic flow are presented. In addition to an active control modification (use of control surface with nonlinear feedback), passive modifications (e.g., a nonlinear damper) are also investigated.

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# LIST OF SYMBOLS

$a$	=transient function, Eq. 39
$ a _{LC} =  \beta_R/\gamma_R ^{1/2}$	=limit cycle value of $ a $
$A(\lambda)$	=matrix of linear terms, Eq. 1
$A^+(\lambda)$	=see Eq. 58
$A_n$	=coefficients of expansion of $A$ , Eq. 16
$b_{npg}$	=coefficients of second-order nonlinear terms, Eq. 2
$c_{npg}$	=coefficients of third-order nonlinear terms, Eq. 2
$f(x, \lambda)$	=vector of nonlinear terms, Eq. 1
$k$	=initial-condition constant, Eq. 39
$t$	=nondimensional time
$u$	=right eigenvector of $A$ , Eq. 8
$u^+$	=see Eq. 62
$v$	=left eigenvector of $A$ , Eq. 29
$v^+$	=see Eq. 64
$x$	=vector of unknowns, Eq. 1
$x^+$	=see Eq. 57
$x_n$	=coefficients of expansion of $x$ , Eq. 18
$\beta$	=see Eq. 32
$\gamma$	=see Eq. 33
$\gamma^+$	=see Eq. 70
$\lambda$	=stability parameter, Eq. 1
$\lambda_n$	=coefficients of expansion of $\lambda$ , Eq. 11
$\tau = \varepsilon^2 t$	=slow time scale
$\tau_0$	= $-(\ln  k )/(2 \beta_R)$
$\omega_0$	=limit cycle frequency, Eq. 4
$\omega_2$	=coefficient of expansion of $\omega$ , Eq. 42

## SECTION I

### INTRODUCTION

The linear description of flutter and the use of linear active control to increase the flutter speed (flutter suppression) are assumed to be known and therefore neither a description of the phenomenon nor a review of the extensive literature in the field are provided here (see, for instance, Refs. 1, 2 and 3). Rather, the phenomenon of interest here is the behavior of an aeroservoelastic system beyond the flutter speed, i. e., in the range for which the linear analysis predicts unstable behavior. The nature of nonlinear flutter was considerably clarified by the research on nonlinear panel flutter which includes the pioneering works of Librescu (Ref. 4) and Kobayashi (Ref. 5) as well as the extensive work of Dowell (Ref. 6 - 10). The results of this research activity indicate that the post-flutter behavior may be characterized as follows: the amplitude of the oscillation of the panel beyond the flutter speed grows in time (in agreement with the instability predicted by the linear analysis) but, in contrast to the linear analysis prediction, the amplitude growth often tends to a finite value; the limiting value solution is called limit cycle. The above phenomenon is called benign flutter. If the amplitude grows beyond any limit, the phenomenon is called destructive flutter. The mechanism of nonlinear panel flutter was clarified from a mathematical point of view by this author in Ref. 11. Because of the considerable bearing on the results presented here, the above work and its extensions are extensively reviewed here. In Ref. 11, an analytic expression of the transient response was obtained in the form of an asymptotic solution using a singular perturbation method (known as multiple time-scaling method) which had been developed, apparently independently, by Sandri (Ref. 12), Cole and Kevorkian (Ref. 13), and Nayfeh (Ref. 14).

The results indicate that in the neighborhood of the instability boundary, an approximate solution (with the order of magnitude of the error equal to the fourth power of the limit cycle amplitude) is obtained for the transient response of the panel; the nature of this solution is characterized by one single real number (indicated with  $\gamma_1$  in Ref. 11). If such number is positive, then the nonlinear terms have a stabilizing effect and they counteract the effect of the linear terms (which are destabilizing above the flutter speed) to yield benign flutter (stable limit

cycle). On the other hand, if such a number is negative, then destructive flutter occurs: it should be emphasized that in this case even for speeds of flight below the flutter speed, where the linear analysis predicts stable behavior, the amplitude may grow beyond any limit provided that the initial amplitude is "sufficiently" high: from a mathematical point of view this behavior is characterized by the presence of an unstable limit cycle. The mathematical description of nonlinear flutter is presented in detail later in this paper.

It should be emphasized that the results of Ref. 11 are not limited to panel flutter: as shown in Ref. 15, the results apply to any dynamic system which can be represented by a system of nonlinear differential equations with nonlinear terms of analytic nature. In addition, the analysis can be extended to include the effect of fifth order nonlinearities (Ref. 16). Finally, the same results are obtained by using a completely independent technique, the Lie transform method (Ref. 17).

Two slightly different formulations of flutter analysis are used in Refs. 11 and 15, respectively. The first one, presented in Ref. 11 deals with dynamic systems which can be represented by a system of second order differential equations. This formulation (presented in terms used for structural dynamics problems) is convenient to study for instance flutter taming of an aircraft that is not subject to linear active control (in particular, without flutter suppression): such a system will be referred to as an aeroelastic system, in contrast to a aeroservoelastic system, which includes active control. Since the equations governing active control are usually given as a set of first order differential equations (state-variable approach), in order to study flutter and/or flutter taming of an aircraft subject to active control, it is convenient to recast the mathematical model as a set of first order differential equations. This system is analyzed in Ref. 15 (where the equivalence of the two formulations is also discussed). This second approach is used here because of its generality and its formal simplicity.

The objective of this work is to show (via a singular-perturbation analysis) that nonlinear feedback control may be used to "tame" the phenomenon of flutter, that is to

ensure that only benign flutter (stable limit cycle) occur and that the limit-cycle amplitude be below any prescribe value. The general theory of Ref. 15 is briefly reviewed in Section 2 and used to discuss flutter taming in Section 3. The methodology is then applied in Section 4 to the particular case of an airfoil in supersonic flow. Numerical results, presented in Section 5, indicate that acceptable values for the limit-cycle amplitude may be obtained. Concluding remarks are presented in Section 6.

## SECTION II

### NONLINEAR FLUTTER

#### 2.1 Introduction

Consider an aeroservoelastic system which may be described by a set of equations of the type

$$d\mathbf{x}/dt = \mathbf{A}(\lambda)\mathbf{x} + \mathbf{f}(\mathbf{x}, \lambda) \quad (1)$$

where  $\mathbf{x}$  is the vector of the state variables describing the systems (e.g., generalized Lagrangian coordinates and velocities as well as control variables). In addition  $\mathbf{A}(\lambda)$  is a square matrix,  $\lambda$  is a positive parameter (e.g., the speed of flight or the dynamic pressure) and  $\mathbf{f}$  is the vector comprising all the nonlinear terms which is assumed to be an analytic function of  $\mathbf{x}$  so that

$$\mathbf{f} = \left\{ \sum_{p,q} b_{pq}(\lambda) x_p x_q + \sum_{p,q,r} c_{pq,r}(\lambda) x_p x_q x_r + \dots \right\} \quad (2)$$

The assumption under which Eq. 1 provides a satisfactory representation of a typical aeroelastic system are examined in Appendix A. For very small amplitudes the system is approximated by the linearized equation

$$d\mathbf{x}/dt = \mathbf{A}(\lambda)\mathbf{x} \quad (3)$$

It is assumed that, for  $\lambda = 0$ , the system is stable, in the sense that all the eigenvalues,  $s$ , of the matrix  $\mathbf{A}(0)$  have a negative real part. It is also assumed that as  $\lambda$  increases, at  $\lambda = \lambda_0$ , one (and only one) pair of complex conjugate eigenvalues of  $\mathbf{A}(\lambda)$  cross the imaginary axis. Let

$$s = i\omega_0 \quad (4)$$

be such an eigenvalue:  $\omega_0$  is called the flutter frequency whereas  $\lambda_0$  is called the flutter boundary or stability boundary. Summarizing, we may say that for  $0 \leq \lambda < \lambda_0$ , all the eigenvalues have a negative real part. For  $\lambda_0 < \lambda < \lambda'_0$  (where  $\lambda'_0$  is the value of  $\lambda$  at which another crossing of the imaginary axis occurs), one pair of complex conjugate eigenvalues has a positive real part and the others have a negative real part. For  $\lambda = \lambda_0$ , one pair of complex conjugate

parts. Recall that (assuming for simplicity that the eigenvalues are distinct) the solution of the linear system is of the type

$$\underline{x} = \sum_i c_i e^{s_i t} \underline{u}_i \quad (5)$$

where  $c_i$  are constants (which depend upon the initial conditions), whereas  $s_i$  and  $\underline{u}_i$  are the eigenvalues and the corresponding eigenvectors of the matrix  $\underline{A}(\lambda)$ , that is, by definition, are such that

$$[s_i \underline{I} - \underline{A}(\lambda)] \underline{u}_i = \underline{0} \quad (6)$$

with  $|\underline{u}_i| \neq 0$ . Then we may say that for  $0 \leq \lambda < \lambda_0$  the solution of the linear system is stable, for  $\lambda_0 < \lambda < \lambda_0'$  it is unstable, whereas for  $\lambda = \lambda_0$  the solution is at the stability boundary: as time goes to infinity, the contribution of the negative-real-part eigenvalues goes to zero and therefore the steady state solution is

$$\underline{x} = c e^{i\omega_0 t} \underline{u} + c^* e^{-i\omega_0 t} \underline{u}^* \quad (7)$$

where \* indicates the complex conjugate value,  $c$  is a constant which depends upon the initial conditions,  $i\omega_0$  is the imaginary eigenvalue and  $\underline{u}$  is the corresponding eigenvector so that

$$[i\omega_0 \underline{I} - \underline{A}_0] \underline{u} = \underline{0} \quad (8)$$

with

$$\underline{A}_0 = \underline{A}(\lambda_0) \quad (9)$$

In essence, the multiple-time-scale analysis used in Refs. 11 and 15 consists in finding the solution for values of  $\lambda$  in the neighborhood of  $\lambda_0$ , using a singular perturbation of the steady-state solution at  $\lambda = \lambda_0$  (given by Eq. 7). This analysis is presented in simplified form in the following subsections.

Two cases are not included here. The first is the case in which at  $\lambda = \lambda_0$  a real eigenvalue (instead of a pair of complex conjugate eigenvalues) crosses the imaginary axis. In aeroelasticity, this instability is known as divergence. This case is actually simpler than that discussed here and is analyzed in Appendix F of Ref. 15. The second case is

one pair of complex conjugate eigenvalues. This case is of little practical interest here and to this author's knowledge has never been analyzed from an analytical point of view.

## 2.2 Nonlinear Flutter Solution

For the sake of simplicity, we will assume that the second-order nonlinear terms are equal to zero (this corresponds for instance to a "symmetric" system, i.e., a structure that is invariant to changes in sign of the state variables). As shown in Refs. 11 and 13, the case of nonzero second-order nonlinear terms adds complexity to the analysis without adding any new feature to the phenomenon. An approximate solution of Eq. 1 can be obtained as follows. Let

$$\varepsilon = |\lambda - \lambda_0|^{1/2} \quad (10)$$

so that

$$\lambda = \lambda_0 + \varepsilon^2 \lambda_2 \quad (11)$$

with

$$\lambda_2 = \pm 1 \quad (12)$$

and assume that the solution for  $\underline{x}$  is a function of time as follows:

$$\underline{x} = \underline{x}(t, \tau) \quad (13)$$

where

$$\tau = \varepsilon^2 t \quad (14)$$

This means that time dependence appears in two different forms: the first one is independent of  $\varepsilon$ , while the second one, the so-called slow-scale, becomes slower as  $\varepsilon$  goes to zero. On the other hand, introducing the transformation given by Eq. 14, the dependence of  $\underline{x}$  upon  $\tau$  is independent of  $\varepsilon$ , whereas, the dependence upon the fast scale,  $\tau/\varepsilon^2$ , becomes faster and faster as  $\varepsilon$  goes to zero. Note that  $t$  and  $\tau$  correspond to the variables  $t_1$  and  $t_2$  in Refs. 11, 12, 14 and 15. Because we are limiting our analysis to  $O(\varepsilon^2)$ , only these two time scales are needed. In this sense, the method used here (multiple time scales) is different from the two-time-scale method used, for instance, by Cole and Kevorkian (Ref. 13).

Equation 13 implies that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial \tau} \quad (15)$$

We will also assume that  $\underline{A}$  and  $c_{npqr}$  are analytic functions of  $\lambda$  so that

$$A(\lambda) = A_0 + \varepsilon^2 \lambda_2 A_2 + O(\varepsilon^4) \quad (16)$$

where  $A_0 = A(\lambda_0)$  and  $A_2 = \partial A / \partial \lambda|_{\lambda=\lambda_0}$ , whereas

$$c_{npqr}(\lambda) = c_{npqr}(\lambda_0) + O(\varepsilon^2) \quad (17)$$

Consider a solution of the type

$$\underline{x} = \varepsilon \underline{x}_1 + \varepsilon^3 \underline{x}_3 + O(\varepsilon^5) \quad (18)$$

where  $\underline{x}_i$  are functions of  $t$  and  $\tau$ .

Combining Eqs. 1 and 15-18, yields

$$\begin{aligned} & \varepsilon \left[ \partial \underline{x}_1 / \partial t - \underline{A}_0 \underline{x}_1 \right] \\ & + \varepsilon^3 \left[ \partial \underline{x}_3 / \partial t + \partial \underline{x}_1 / \partial \tau - \underline{A}_0 \underline{x}_3 - \lambda_2 \underline{A}_2 \underline{x}_1 + \underline{f}_3 \right] \\ & + O(\varepsilon^5) = 0 \end{aligned} \quad (19)$$

where all the terms in brackets are not explicitly functions of  $\varepsilon$  (it may be noted that this is the mathematical reason for introducing the slow time scale  $\varepsilon$ ) and

$$\underline{f}_3 = \left\{ \sum_{p,q,r} c_{npqr}(\lambda_0) x_{1,p} x_{1,q} x_{1,r} \right\} \quad (20)$$

Separating terms of the same order of magnitude yields

$$\partial \underline{x}_1 / \partial t - \underline{A}_0 \underline{x}_1 = 0 \quad (21)$$

$$\partial \underline{x}_3 / \partial t - \underline{A}_0 \underline{x}_3 = -\partial \underline{x}_1 / \partial \tau + \lambda_2 \underline{A}_2 \underline{x}_1 + \underline{f}_3 \quad (22)$$

Disregarding the terms in the solution corresponding to characteristic roots with real parts, the solution of Eq. 22 is

$$\underline{x}_1 = a(\tau) \underline{u} e^{i\omega_0 t} + a^*(\tau) \underline{u} e^{-i\omega_0 t} \quad (23)$$

where  $i\omega_0$  is the imaginary eigenvalue of  $\underline{A}_0$  and  $\underline{u}$  is the corresponding eigenvector (see Eq. 8): the main difference



between the solution of the linear problem, Eq. 7, and the first order solution of the nonlinear problem, Eq. 23, is that  $c$  is a constant whereas, in general,  $a$  is a function of  $\tau$  (note that in Eq. 21 the time derivative is a partial derivative, whereas in the linear problem, Eq. 3, the time derivative is an ordinary derivative).

Substituting Eq. 23 into Eq. 22 and noting that

$$\begin{aligned} \underline{f}_3 &= \left\{ \sum_{pqr} c_{npqr}(\lambda_0) (a u_p e^{i\omega_0 t} + CCT) \right. \\ &\quad \left. (a u_q e^{i\omega_0 t} + CCT) (a u_r e^{i\omega_0 t} + CCT) \right\} \\ &= \underline{f}_3^{(1)} a^3 e^{i3\omega_0 t} + \underline{f}_3^{(1)} a^2 a^* e^{i\omega_0 t} + CCT \end{aligned} \quad (24)$$

where CCT indicates complex conjugate terms and

$$\underline{f}_3^{(2)} = \left\{ \sum_{pqr} c_{npqr}(\lambda_0) u_p u_q u_r \right\} \quad (25)$$

$$\underline{f}_3^{(1)} = \left\{ \sum_{pqr} c_{npqr}(\lambda_0) (u_p u_q u_r + u_p u_q u_r + u_p u_q u_r) \right\} \quad (26)$$

one obtains

$$\begin{aligned} \partial \underline{x}_3 / \partial t - \underline{A}_0 \underline{x}_3 &= a^3 \underline{f}_3^{(2)} e^{i3\omega_0 t} \\ &+ \left[ -\underline{u} \partial a / \partial \tau + \lambda_2 \underline{A}_2 \underline{u} a + \underline{f}_3^{(1)} a^2 a^* \right] e^{i\omega_0 t} \\ &+ CCT \end{aligned} \quad (27)$$

The solution contains "secular terms" (i.e., terms of the type  $t e^{i\omega_0 t}$ ) unless

$$\underline{u}^T \left[ -\underline{u} \partial a / \partial \tau + \lambda_2 \underline{A}_2 \underline{u} a + \underline{f}_3^{(1)} a^2 a^* \right] = 0 \quad (28)$$

where  $\underline{v}^T$  is the left eigenvector of  $\underline{A}_0$  corresponding to the eigenvalue  $i\omega_0$ , i.e., the nontrivial solution of

$$\underline{v}^T [i\omega_0 \underline{I} - \underline{A}_0] = \underline{0} \quad (29)$$

If  $\underline{v}$  is normalized so that

$$\underline{v}^T \underline{u} = 1 \quad (30)$$

Eq. 28 may be written as

$$\frac{\partial a}{\partial \tau} + \beta a + \gamma a^2 a^* = 0 \quad (31)$$

where

$$\beta = -\lambda_2 \underline{v}^T \underline{A}_2 \underline{u} \quad (32)$$

$$\gamma = -\underline{v}^T \underline{f}_3^{(1)} \quad (33)$$

By setting

$$a = |a| e^{i\varphi} \quad (34)$$

and separating real and imaginary parts one obtains

$$\partial |a| / \partial \tau + \beta_R |a| + \gamma_R |a|^3 = 0 \quad (35)$$

$$\partial \varphi / \partial \tau + \beta_I + \gamma_I |a|^2 = 0 \quad (36)$$

Equation 35 may be easily solved by setting  $|a| = 1/\sqrt{z}$  to obtain

$$\partial z / \partial \tau - 2\beta_R z - 2\gamma_R = 0 \quad (37)$$

or

$$z = -\frac{\gamma_R}{\beta_R} (1 + k e^{2\beta_R \tau}) \quad (38)$$

(where  $k$  depends upon the initial condition) and hence

$$|a| = \left| \frac{-\beta_R / \gamma_R}{1 + k e^{2\beta_R \tau}} \right|^{1/2} \quad (39)$$

Substituting into Eq. 36 and integrating yields

$$\begin{aligned} \varphi &= -\int_0^{\tau} (\beta_I + \gamma_I |a|^2) d\tau + \varphi_0 \\ &= (\gamma_I \beta_R / \gamma_R - \beta_I) \tau + (\gamma_I / \gamma_R) \ln |a| + \varphi_0 \end{aligned} \quad (40)$$

Combining Eqs. 10, 14, 18, 23, 34, 39 and 40 one obtains

$$\underline{x} = 2 \operatorname{Re} a l \left( |\lambda - \lambda_0|^{1/2} |a| e^{i(\omega t + \varphi_1)} \underline{u} \right) + O(\varepsilon^3) \quad (41)$$

with

$$\omega = \omega_0 + \varepsilon^2 \omega_2 = \omega_0 + |\lambda - \lambda_0| \omega_2 \quad (42)$$

(where  $\omega_2 = \gamma_I \beta_R / \gamma_R - \beta_I$ ) and

$$\varphi_1 = (\gamma_I / \gamma_R) \ln |a| + \varphi_0 \quad (43)$$

whereas  $i\omega_0$  is the imaginary eigenvalue (and  $\underline{u}$  the corresponding eigenvector) of  $A_0$  (see Eqs. 8 and 9) and  $|a|$  is given by Eq. 39.

### 2.3 Discussion of Solution

The function  $|a|$  divided by  $|\beta_R / \gamma_R|^{1/2}$ , i. e., (see Eq. 39)

$$\frac{|a|}{|\beta_R / \gamma_R|^{1/2}} = \frac{-\operatorname{sgn}(\beta_R / \gamma_R)}{1 - \operatorname{sgn} k e^{2\beta_R(\tau - \tau_0)}} \quad (44)$$

where

$$\tau_0 = -\ln |k| / 2\beta_R \quad (45)$$

is plotted in Figs. 1-4 for all of the possible combinations of the signs of  $\beta_R$  and  $\gamma_R$ . Only real values of  $|a|$  are plotted. It is worth noting that the absolute value of  $k$  influences only the value of  $\tau_0$ , which in turn determines the value of the abscissa that corresponds to the time origin  $\tau = 0$ . Note that (see Eq. 32)

$$\beta_R \leq 0 \quad \text{for} \quad \lambda_2 = \pm 1 \quad (\lambda \geq \lambda_0) \quad (46)$$

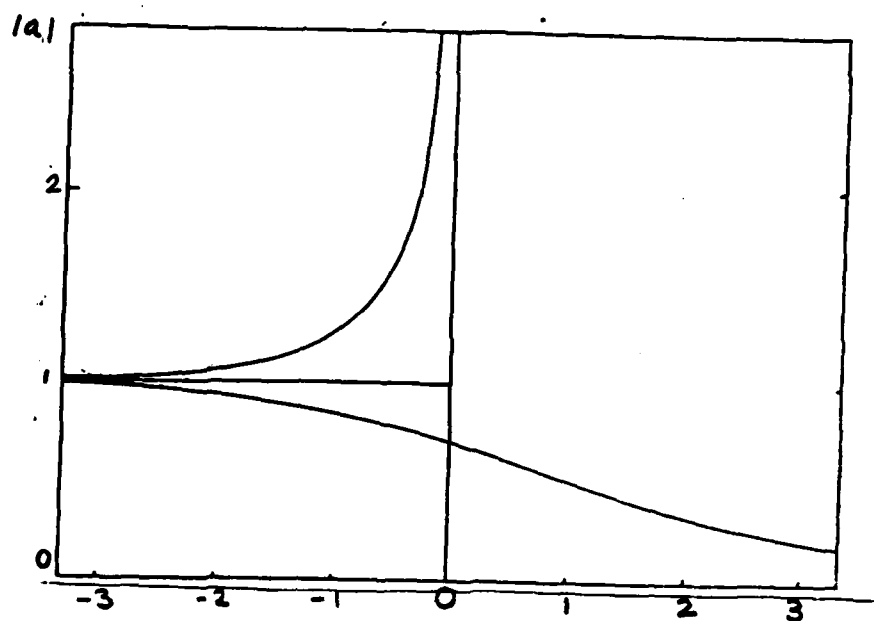


Figure 1

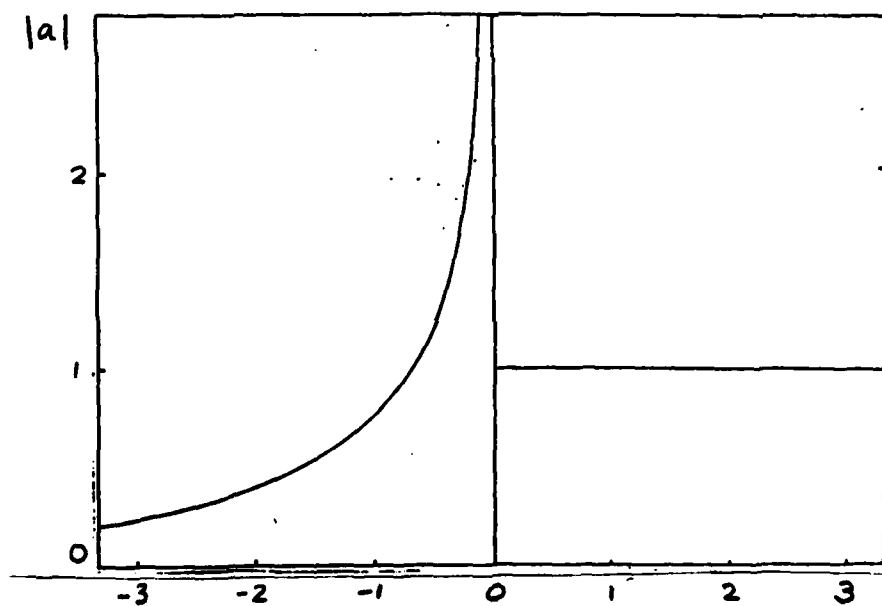


Figure 2

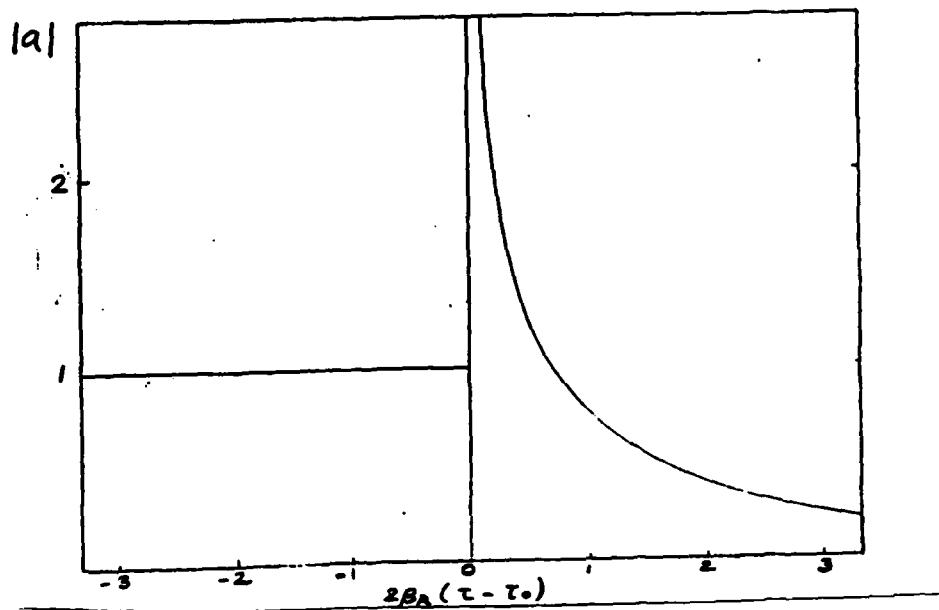


Figure 3

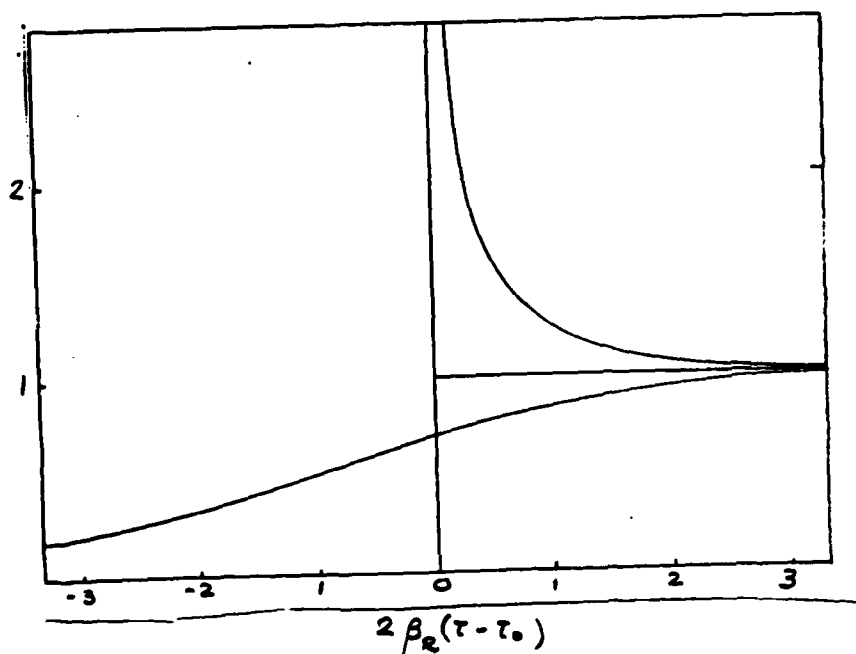


Figure 4

since the linear system is unstable (i.e.,  $\beta_R < 0$ ) for  $\lambda_2 = 1$  (i.e.,  $\lambda > \lambda_0$ ). Consider, separately the cases  $\gamma_R > 0$  and  $\gamma_R < 0$ .

### Case 1 ( $\gamma_R > 0$ )

Consider first Fig. 1, which corresponds to  $\gamma_R > 0$  (stabilizing nonlinear terms) and  $\lambda > \lambda_0$  (i.e.,  $\beta_R < 0$ , destabilizing linear terms). In this case, there exist two branches, depending upon the sign of  $k$ . If the initial conditions are such that at time  $\tau = 0$ ,  $|a| > |\beta_R / \gamma_R|$ , then  $k < 0$  and the amplitude decreases to the limit-cycle value

$$\lim_{\tau \rightarrow \infty} |a| = |a|_{LC} = |\beta_R / \gamma_R|^{1/2} \quad (47)$$

Viceversa if  $|a| < |\beta_R / \gamma_R|^{1/2}$ , then  $k > 0$  and the amplitude increases to the same limit-cycle value. In other words, for  $\gamma_R > 0$  and  $\lambda > \lambda_0$  (i.e.,  $\beta_R < 0$ ), there exists a stable limit cycle; note that the solution  $|a| = 0$  is unstable. This behavior could have been predicted from Eq. 31 by noting that for large values of  $|a|$  the stabilizing nonlinear terms are dominant, whereas for small values of  $|a|$  the destabilizing linear terms are dominant.

Next consider Fig. 2, which corresponds to  $\gamma_R > 0$  (stabilizing nonlinear terms) and  $\lambda < \lambda_0$  (i.e.,  $\beta_R > 0$ , stabilizing linear terms). In this case, real values for  $|a|$  are obtained only if  $k < 0$  and  $\tau > \tau_0$ . The solution tends to zero:

$$\lim_{\tau \rightarrow \infty} |a| = 0 \quad (48)$$

i.e., the solution  $|a| = 0$  is stable. This behavior is to be expected, since both linear and nonlinear terms in Eq. 31 are stabilizing.

Summarizing, if  $\gamma_R > 0$  (stabilizing nonlinear terms), there exists a stable limit cycle for  $\lambda > \lambda_0$  (when the linear analysis predicts instability), whereas for  $\lambda < \lambda_0$  the solution  $|a| = 0$  is unconditionally stable.

### Case 2 ( $\gamma_R < 0$ )

Next consider Fig. 3, which corresponds to  $\gamma_R < 0$  (destabilizing nonlinear terms) and  $\lambda > \lambda_0$  (i.e.,  $\beta_R < 0$ ,

destabilizing linear terms). In this case,  $|a|$  is real only for  $k < 0$  and  $\tau < \tau_0$ . The solution tends to infinity as  $\tau$  tends to  $\tau_0$ :

$$\lim_{\tau \rightarrow \tau_0} |a| = \infty \quad (49)$$

i.e., the solution is unconditionally unstable. This behavior is predictable since both linear and nonlinear terms in Eq. 31 are destabilizing.

Last consider Fig. 4, which corresponds to  $\gamma_R < 0$  (destabilizing nonlinear terms) and  $\lambda < \lambda_0$  (i.e.,  $\beta_R > 0$ , stabilizing linear terms). Note that two branches exist, for  $k \geq 0$ , respectively. In both cases

$$\lim_{\tau \rightarrow -\infty} |a| = |a|_{LC} = |\beta_R / \gamma_R|^{1/2} \quad (50)$$

However, if the initial conditions are such that  $|a| > |a|_{LC}$  at time  $\tau = 0$  (i.e.,  $k < 0$ ), then the solution grows to infinity:

$$\lim_{\tau \rightarrow \tau_0} |a| = \infty \quad \text{if } |a|_0 > |a|_{LC} \quad (51)$$

On the other hand, if the initial conditions are such that  $|a| < |a|_{LC}$  at time  $\tau = 0$  (i.e.,  $k > 0$ ), then the solution goes to zero:

$$\lim_{\tau \rightarrow \infty} |a| = 0 \quad \text{if } |a|_0 < |a|_{LC} \quad (52)$$

The preceding behavior may be characterized by saying that there exists an unstable limit cycle. The solution  $|a| = 0$  is stable, whereas the solution  $|a| = |a|_{LC}$  is unstable. In other words, the system is conditionally stable (i.e., is stable only if the initial conditions are sufficiently small, whereas it is unstable if the initial conditions are sufficiently large). This behavior could have been predicted from Eq. 31 by noting that the stabilizing linear terms dominate for small values of  $|a|$ , whereas destabilizing nonlinear terms dominate for large values of  $|a|$ . It should be emphasized that this behavior is very significant in practical applications, since, if the initial conditions are sufficiently large, instability occurs even if the linear analysis predicts stability. The primary objective of flutter taming is to prevent this phenomenon.

Summarizing, if  $\gamma_R < 0$  (destabilizing nonlinear terms), there exists an unstable limit cycle for  $\lambda < \lambda_0$  (when the linear analysis predicts stability), whereas for  $\lambda > \lambda_0$  the solution is unconditionally unstable.

In conclusion, the results of this section may be restated as follows: there always exists a limit cycle given by (see Eq. 41)

$$\underline{x} = 2 |(\lambda - \lambda_0) \beta_R / \gamma_R|^{1/2} \text{Real} (\underline{u} e^{i(\omega t + \varphi_*)}) + O(\epsilon^3) \quad (53)$$

where  $\varphi_* = (\gamma_I / \gamma_R) \ln |\beta_R / \gamma_R|^{1/2} + \varphi_0$  is a constant. If  $\gamma_R > 0$  (stabilizing nonlinear terms), a stable limit cycle exists for  $\lambda > \lambda_0$  (with amplitude growing like  $\sqrt{\lambda - \lambda_0}$ ), whereas the system is stable for  $\lambda < \lambda_0$ . This behavior is indicated in Fig. 5, where the arrows indicate the variation with time. Viceversa if  $\gamma_R < 0$  (destabilizing nonlinear term), an unstable limit cycle exists for  $\lambda < \lambda_0$  (with amplitude growing like  $\sqrt{\lambda - \lambda_0}$ ) whereas the system is unstable for  $\lambda > \lambda_0$ . This behavior is indicated in Fig. 6.



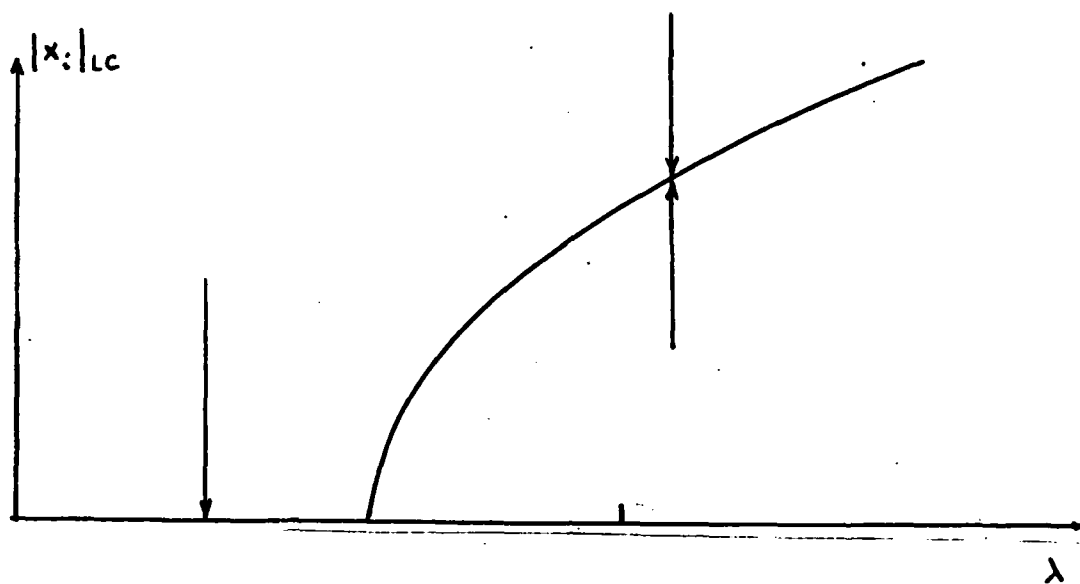


Figure 5

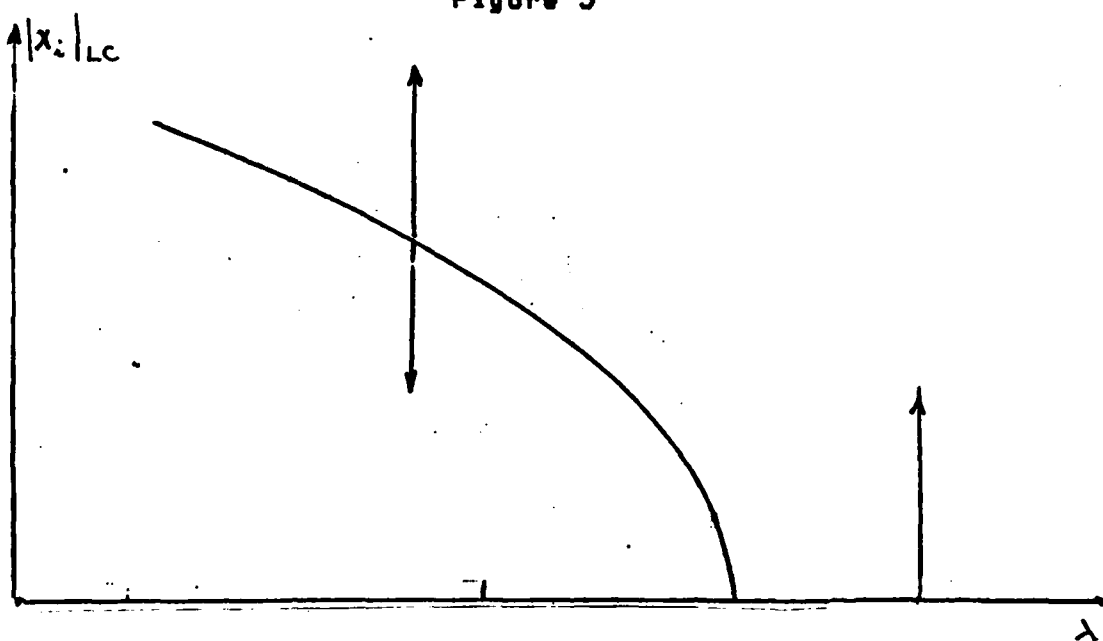


Figure 6

## SECTION III

### THEORY OF FLUTTER TAMING

#### 3.1 Introduction

The results of the analysis of nonlinear flutter presented in Section II indicate that the behavior of the aeroelastic system beyond the flutter boundary (e.g., benign versus destructive flutter) is determined by the sign of the real part of a complex number  $\gamma$  defined by Eq. 33. This suggests a simple procedure for flutter taming: modify the system so that the additional contributions produce the desired positive value for  $\gamma_R$ . It is convenient to add the requirement that the linear characteristics of the modified system be equal to those of the original system. This is important in order to avoid, for instance, that the introduction of flutter taming decrease the flutter speed.

It may be worth recalling that the objective of flutter suppression by active control is to modify the linear system to achieve an increase in the flutter speed (Ref. 3) the above requirement that the linear characteristics before and after the flutter-taming modifications ensures that flutter taming may be made on a system already modified for flutter suppression without affecting the flutter-suppression modifications.

Two types of modifications are considered here. In the first one (type A), the nonlinear terms of the system are modified to ensure the benign-flutter condition. These modifications could be a modification of the nonlinear elastic characteristics or the addition of a nonlinear damper. In the second one (type B), new state variables (typically active control) with a nonlinear feedback are added to the system (augmented system). This can be accomplished, for instance, by having a sensor (e.g., an accelerometer) and an actuator which controls a control surface, with a driving force proportional to the cube of a linear combination of the original state variables.

Whereas many different modifications may be included, only the above two types are considered here. The modified system is then described by

$$d\underline{x}/dt = \underline{A}(\lambda) \underline{x} + \underline{f}(\underline{x}, \lambda) + \kappa' \underline{f}'(\underline{x}, \lambda) + \underline{B}(\lambda) \underline{y} \quad (54)$$

$$d\underline{y}/dt = \underline{D} \underline{y} + \kappa'' \underline{z} (\underline{c}^T \underline{x})^3 \quad (55)$$

The term  $\kappa' \underline{f}'(\underline{x}, \lambda)$  (where  $\kappa'$  is a constant and  $\underline{f}'$  is a nonlinear vector) corresponds to the type-A modification. On the other hand, the term  $\underline{B}\underline{y}$  (with  $\underline{y}$  governed by Eq. 55) corresponds to type-B modification. The nonlinear feedback is provided by the term  $\kappa'' \underline{z} (\underline{c}^T \underline{x})^3$  (where  $\underline{z}$  and  $\underline{c}$  are arbitrary vectors and  $\kappa''$  is an arbitrary constant). Note that (as shown in details later), the lack of linear feedback in Eq. 55 guarantees that the linear characteristics of the modified system are equal to those of the original system. Note also that by setting  $\kappa' = 0$  one obtains the modification of type B, whereas setting  $\underline{B} = 0$  one recovers the modification of type A. For the sake of convenience, the analysis is carried out under the assumption that both modifications are present at the same time.

### 3.2 Flutter Taming Solution

Equations 54 and 55 may be recast as

$$d\underline{x}^+/dt = \underline{A}^*(\lambda) \underline{x}^+ + \underline{f}^*(\underline{x}^+, \lambda) \quad (56)$$

where

$$\underline{x}^+ = \begin{Bmatrix} \underline{x} \\ \underline{y} \end{Bmatrix} \quad (57)$$

$$\underline{A}^*(\lambda) = \begin{bmatrix} \underline{A}(\lambda) & \underline{B}(\lambda) \\ \underline{0} & \underline{D} \end{bmatrix} \quad (58)$$

$$\underline{f}^*(\underline{x}^+, \lambda) = \begin{Bmatrix} \underline{f} + \kappa' \underline{f}' \\ \kappa'' \underline{z} (\underline{c}^T \underline{x})^3 \end{Bmatrix} \quad (59)$$

Equation 56 is formally equal to Eq. 1 and therefore the results of Section II are applicable to Eq. 56 as well. Hence, the solution of the linear system is obtained by setting

$$\det \begin{bmatrix} s \underline{I} - \underline{A} & -\underline{B} \\ 0 & s \underline{I} - \underline{D} \end{bmatrix} = \det [s \underline{I} - \underline{A}] \det [s \underline{I} - \underline{D}] = 0 \quad (60)$$

which shows that the eigenvalues of the modified system are equal to those of the original system plus those of the actuator system.

In particular, for  $\lambda = \lambda_0$ , an eigenvalue,  $s = i\omega_0$ , is on the imaginary axis and (assuming that the actuator system does not have an eigenvalue equal to  $i\omega_0$ ) the corresponding eigenvector  $\underline{u}^+$ , i.e., a nontrivial solution of

$$\begin{aligned} [i\omega_0 \underline{I} - \underline{A}_0] \underline{u}^+ \\ = \begin{bmatrix} i\omega_0 \underline{I} - \underline{A}_0 & -\underline{B}_0 \\ 0 & i\omega_0 \underline{I} - \underline{D} \end{bmatrix} \underline{u}^+ = 0 \end{aligned} \quad (61)$$

(where  $\underline{A}_0 = \underline{A}(\lambda_0)$  and  $\underline{B}_0 = \underline{B}(\lambda_0)$ ) is given by

$$\underline{u}^+ = \begin{Bmatrix} \underline{u} \\ 0 \end{Bmatrix} \quad (62)$$

(where  $\underline{u}$  is the eigenvector of  $\underline{A}_0$ , defined by Eq. 9). In addition, the left eigenvector,  $\underline{v}^+$ , i.e., the solution of

$$\begin{aligned} \underline{v}^{+T} [i\omega_0 \underline{I} - \underline{A}_0] \\ = \underline{v}^{+T} \begin{bmatrix} i\omega_0 \underline{I} - \underline{A}_0 & -\underline{B}_0 \\ 0 & i\omega_0 \underline{I} - \underline{D} \end{bmatrix} = 0 \end{aligned} \quad (63)$$

is given by

$$\underline{v}^+ = \begin{Bmatrix} \underline{v} \\ \underline{\tilde{v}} \end{Bmatrix} \quad (64)$$

where  $\underline{v}$  is the left eigenvector of  $\underline{A}_0$  and

$$\underline{\ddot{x}}^T = \underline{v}^T \underline{B} [i\omega_0 \underline{I} - \underline{D}]^{-1} \quad (65)$$

The condition for benign flutter is given by

$$\gamma_R^+ = \text{Real } \gamma^+ > 0 \quad (66)$$

with

$$\gamma^+ = \underline{v}^{+T} \underline{f}_3^{(+)} \quad (67)$$

where

$$\underline{f}_3^{(+)} = \left\{ \begin{array}{l} \underline{f}_3^{(+)} + \kappa' \underline{f}_3^{(+)} \\ \kappa'' \underline{z} \pm 3\mu^2 \mu^* \end{array} \right\} \quad (68)$$

with  $\underline{f}_3^{(+)}$  given by Eq. 26 and  $\underline{f}_3^{(-)}$  given by a similar expression, whereas

$$\mu = \underline{c}^T \underline{u} \quad (69)$$

Using Equations 64 and 68, Eq. 67 may be written as

$$\gamma^+ = \gamma + \kappa' \gamma' + \kappa'' \gamma'' \quad (70)$$

where

$$\gamma = \underline{v}^T \underline{f}_3^{(+)} \quad (71)$$

is due to the nonlinearities of the original system

$$\gamma' = \underline{v}^T \underline{f}_3^{(1)} \quad (72)$$

is due to the additional nonlinearities of the control surface aerodynamics and

$$\gamma'' = 3\mu^2 \mu^* \underline{v}^T \underline{z} \quad (73)$$

The condition for benign flutter is

$$\gamma_R^+ = \gamma_R + \kappa' \gamma_R' + \kappa'' \gamma_R'' \quad (74)$$

which indicates that if  $\gamma_R' \neq 0$  (or  $\gamma_R'' \neq 0$ ), it is always possible to choose  $\kappa'$  (or  $\kappa''$ ) to satisfy the above condition. Note that it is easy to ensure that  $\gamma_R'' \neq 0$ . For, it is always possible to choose  $\underline{c}$  such that

$$|\mu| = |\underline{c}^T \underline{u}| \neq 0 \quad (75)$$

and to choose  $z$  such that

$$\gamma_R'' = |\mu|^2 \operatorname{Real} (\mu \check{v}^T \check{z}) \neq 0 \quad (76)$$

As a final remark, note that as  $\kappa' \operatorname{sgn}(\gamma_R')$  goes to infinity,  $\gamma_R^+$  goes to infinity and hence the limit-cycle amplitude goes to zero (similarly for  $\kappa''$ ). In other words, the amplitude of the limit-cycle can be made as small as desired by choosing appropriate values for the constants  $\kappa'$  and  $\kappa''$ .

## SECTION IV

### AIRFOIL IN SUPERSONIC FLOW

#### 4.1 Governing Equations

Consider an airfoil in a supersonic stream (see Fig. B.1). Let the airfoil have only two degrees of freedom: the vertical displacement,  $h = b\zeta$  (where  $b$  is the semichord), of the center of mass and the rotation,  $\alpha$ , around the center of mass. Let the airfoil be restrained by two springs at a point E (representing the elastic axis of a wing). The first spring produces a vertical force, function of the vertical displacement at E. The second produces a moment function of  $\alpha$ . Let the two springs be of symmetric nature (e.g., the same force is obtained in tension and compression) and defined by the constants  $K_1, K_2, K'_1$ , and  $K'_2$ , see Eq. B.41. Assume that the piston theory gives a sufficiently accurate description for the pressure distribution. Then, as shown in Appendix B, the governing equations are given by Eq. B.51 which may be recast as Eq. 1 with

$$\underline{x}^T = [\zeta \quad \alpha \quad \dot{\zeta} \quad \dot{\alpha}] \quad (77)$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_{11} & -k_{12} & -\sigma S_0 & \sigma S_1 \\ \frac{-k_{12}}{\nu} & \frac{-k_{22}}{\nu} & \frac{\sigma S_1}{\nu} & \frac{-\sigma S_2}{\nu} \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & S_0 & 0 & 0 \\ 0 & \frac{-S_1}{\nu} & 0 & 0 \end{bmatrix} \quad (78)$$

and

$$\underline{f} = \begin{Bmatrix} 0 \\ 0 \\ g_{\zeta}^{(s)} + g_{\zeta}^{(A)} \\ \frac{g_{\alpha}^{(s)} + g_{\alpha}^{(A)}}{\nu} \end{Bmatrix} \quad (79)$$

In Eq. 78  $\nu$  is the nondimensional moment of inertia (see Eq. B.19),  $k_{ij}$  are constants (given by Eqs. B.48 and B.44 in terms of the spring constants, the speed of sound  $a_\infty$  and the mass  $m$ ),  $S_i$  are constants given by the Eq. B.33 in the terms of  $\xi_0$  (nondimensional location of midchord with respect to

the center of mass). In addition  $\sigma$  is the mass ratio (given by Eq. B.27) and

$$\lambda = M\sigma = \frac{U_\infty}{a_\infty} \frac{4\rho_\infty b^2}{m} \quad (80)$$

(where  $M=U_\infty/a_\infty$  is the Mach number) is proportional to the speed of flight.

In addition the structural nonlinear terms are given by (see Eqs. B.44, B.49 and B.50)

$$\begin{aligned} g_S^{(3)}(\underline{x}) &= \frac{1}{6} k_S \delta \alpha^3 - k'_S (\zeta + \delta \alpha)^3 \\ g_\alpha^{(6)}(\underline{x}) &= -\frac{1}{2} k_S \delta \zeta \alpha^2 + \left(\frac{2}{3} k_S \delta^2 - k'_\alpha\right) \alpha^3 - k'_S \delta (\zeta + \delta \alpha)^3 \end{aligned} \quad (81)$$

Finally the aerodynamic nonlinear terms are given by (see Eqs. B.35, B.36, B.37, B.39 and B.40)

$$\begin{aligned} g_S^{(A)}(\underline{x}) &= \sigma \left( c_{11} S_0 M \alpha^3 + c_{12} S_0 \alpha^2 \dot{\zeta} + c_{13} S_1 \alpha^2 \dot{\alpha} \right) \\ &\quad + \sigma \chi \left( S_0 w^3 + 3 S_1 w^2 \dot{\alpha} + 3 S_2 w \dot{\alpha}^2 + S_3 \dot{\alpha}^3 \right) \\ g_\alpha^{(A)}(\underline{x}) &= \sigma \left( c_{21} S_1 M \alpha^3 + c_{22} S_1 \alpha^2 \dot{\zeta} + c_{23} S_2 \alpha^2 \dot{\alpha} \right) \\ &\quad + \sigma \chi \left( S_1 w^3 + 3 S_2 w^2 \dot{\alpha} + 3 S_3 w \dot{\alpha}^2 + S_4 \dot{\alpha}^3 \right) \end{aligned} \quad (82)$$

with

$$w = M \alpha - \dot{\zeta} \quad (83)$$

whereas (see Eqs. B.35 and B.36)

$$c_{ij} = \begin{bmatrix} -2/3 & 1 & -1/2 \\ -1/6 & -1/2 & 0 \end{bmatrix} \quad (84)$$

These values are obtained by assuming that the downwash used in the piston theory is in the direction of the  $\eta$ -axis. Similar expressions are obtained by Smith and Morino (Ref. 15) by assuming the downwash to be in the direction of the  $y$ -axis: in this case the coefficients are given by (see Eqs. B.39 and B.40)



$$C_{ij} = \begin{bmatrix} -1/6 & -1/2 & -1 \\ -1/3 & 0 & 1/2 \end{bmatrix} \quad (85)$$

#### 4.2 Linear Flutter Analysis

The solution is of the type given by Eq. 5, where  $s_i$  and  $y_i$  are eigenvalues and the corresponding eigenvectors of A. In the case under consideration here, the eigenvalues are the solution of

$$\det[sI - A] = \det \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ k_{11} & k_{12} - \lambda S_0 & s + \sigma S_0 & -\sigma S_1 \\ \frac{k_{12}}{\nu} & \frac{k_{22} + \lambda S_1}{\nu} & \frac{-\sigma S_1}{\nu} & \frac{s + \sigma S_2}{\nu} \end{bmatrix} = 0 \quad (86)$$

or, adding third and fourth columns (multiplied by  $s$ ) to first and second columns respectively and multiplying the fourth row by  $\nu$ ,

$$\det \begin{bmatrix} s^2 + \sigma S_0 s + k_{11} & -\sigma S_1 s + k_{12} - \lambda S_0 \\ -\sigma S_1 s + k_{12} & \nu s^2 - \sigma S_2 s + k_{22} + \lambda S_1 \end{bmatrix} = 0 \quad (87)$$

which yields

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0 \quad (88)$$

with

$$\begin{aligned} a_4 &= \nu \\ a_3 &= \sigma(\nu S_0 + S_2) \\ a_2 &= \nu K_{11} + k_{22} + \lambda S_1 + \sigma^2(S_0 S_2 - S_1^2) \\ a_1 &= \sigma(S_0 k_{22} + S_2 k_{11} - 2 S_1 k_{12}) \\ a_0 &= k_{11} k_{22} - k_{12}^2 + \lambda(S_1 k_{11} + S_0 k_{12}) \end{aligned} \quad (89)$$

At the stability boundary,  $\lambda = \lambda_0$ , a pair of complex conjugate roots are purely imaginary ( $s = \pm i\omega_0$ , see Eq. 4), and, separating real part and imaginary part, one obtains

$$a_4 \omega_0^4 - a_2 \omega_0^2 + a_0 = 0 \quad (90)$$

$$a_3 \omega_0^3 - a_1 = 0 \quad (91)$$

or, from Eq. 91,

$$\omega_o^2 = (S_o k_{22} + S_2 k_{11} - 2S_1 k_{12}) / (\nu S_o + S_2) \quad (92)$$

and, from Eq. 90,

$$\lambda_o = - \frac{\nu \omega_o^4 - (k_{22} + \nu k_{11}) \omega_o^2 + k_{11} k_{22} - k_{12}^2 + \sigma^2 \omega_o^2 (S_o S_2 - S_1^2)}{-\omega_o^2 S_1 + S_1 k_{11} + S_o k_{12}} \quad (93)$$

It may be worth noting that, by using the expression for  $k_{ij}$  and  $S_i$  (Eqs. B.48 and B.33), Eqs. 92 and 93 may be rewritten as

$$\omega_o^2 = [k_3 (\delta - \xi_o)^2 + \frac{1}{3} k_3 + k_\alpha] / (\nu + \frac{1}{3} + \xi_o^2) \quad (94)$$

and

$$\lambda_o = - \frac{(\omega_o^2 - k_3)(k_\alpha - \nu \omega_o^2) + \delta^2 k_3 \omega_o^2 + \sigma^2 \omega_o^2 / 3}{\omega_o^2 \xi_o + (\delta - \xi_o) k_3} \quad (95)$$

in agreement with Ref. 18 and Ref. 15 (where  $k_3 = \Omega_s^2$ ,  $k_\alpha = \nu \Omega_\alpha^2$ ,  $\xi_o = -\eta_c$  and  $\delta - \xi_o = \eta_s$ ; note a misprint in Eq. C16 of Ref. 15 where  $\omega_o^4$  should be  $\omega_o^2$ ).

The eigenvector  $\underline{u}$  (the nontrivial solution of Eq. 8), normalized by setting  $\underline{u}_1 = 1$ , is given by

$$\underline{u} = \begin{Bmatrix} 1 \\ u \\ i\omega_o \\ i\omega_o u \end{Bmatrix} \quad (96)$$

where

$$u = (-\omega_o^2 + i\omega_o \sigma S_o + k_{11}) / (i\omega_o \sigma S_1 + k_{12} - \lambda_o S_o) \\ = (-i\omega_o \sigma S_1 + k_{12}) / (-\nu \omega_o^2 + i\omega_o \sigma S_2 + k_{22} + \lambda_o S_1) \quad (97)$$

Similarly the left eigenvector  $\underline{v}$  is given by

$$\underline{v}^T = \frac{1}{\alpha_o} [v_1 \quad v_2 \quad 1 \quad \nu v] \quad (98)$$

where

$$v = (-\omega_0^2 + i\omega_0 \epsilon S_0 + k_{11}) / (-i\omega_0 \epsilon S_1 + k_{12}) \quad (99)$$

$$= (-i\omega_0 \epsilon S_1 + k_{12} - \lambda_0 S_0) / (-v\omega_0^2 + i\omega_0 \epsilon S_2 + k_{22} + \lambda_0 S_1)$$

and

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} i\omega_0 + \epsilon S_0 & -\epsilon S_1 \\ -\epsilon S_1 & i\omega_0 v + \epsilon S_2 \end{bmatrix} \begin{Bmatrix} 1 \\ v \end{Bmatrix} \quad (100)$$

whereas  $\alpha_0$  is a normalisation constant such that  $\underline{v}^T \underline{u} = 1$  (see Eq. 30), i. e.,

$$\alpha_0 = 2i\omega_0 (1 + uv) + \epsilon [S_0 + (u+v)S_1 + uvS_2] \quad (101)$$

### 4.3 Nonlinear Flutter

Using Eqs. 20 and 24 one obtains, for the specific case under consideration,

$$\underline{f}_3 = \begin{Bmatrix} 0 \\ 0 \\ g_{53} \\ g_{\alpha 3}/v \end{Bmatrix} \quad (102)$$

with

$$\begin{aligned} g_{53} &= g_5(\underline{x}_1) \\ &= 1/6 k_3 \delta (au e^{i\omega_0 t} + CCT)^3 \\ &\quad - k'_5 [a(1+\delta u) e^{i\omega_0 t} + CCT]^3 \\ &\quad + \epsilon c_{11} S_0 M (au e^{i\omega_0 t} + CCT)^3 \\ &\quad + \epsilon c_{12} S_0 (au e^{i\omega_0 t} + CCT)^2 (ai\omega_0 e^{i\omega_0 t} + CCT) \\ &\quad + \epsilon c_{13} S_1 (au e^{i\omega_0 t} + CCT)^2 (au i\omega_0 e^{i\omega_0 t} + CCT) \\ &\quad + \epsilon \chi S_0 (a\tilde{w} e^{i\omega_0 t} + CCT)^3 \\ &\quad + \epsilon \chi 3S_1 (a\tilde{w} e^{i\omega_0 t} + CCT)^2 (au i\omega_0 e^{i\omega_0 t} + CCT) \\ &\quad + \epsilon \chi 3S_2 (a\tilde{w} e^{i\omega_0 t} + CCT) (au i\omega_0 e^{i\omega_0 t} + CCT)^2 \\ &\quad + \epsilon \chi S_3 (au i\omega_0 e^{i\omega_0 t} + CCT)^3 \end{aligned} \quad (103)$$

and

$$\begin{aligned}
 g_{u3} &= g_u(x_1) \\
 &= \frac{1}{2} k_5 \delta (a e^{i\omega_0 t} + CCT) (a u e^{i\omega_0 t} + CCT)^2 \\
 &\quad + (\frac{2}{3} k_5 \delta^2 - k'_u) (a u e^{i\omega_0 t} + CCT)^3 \\
 &\quad - k'_5 \delta [a(1 + \delta u) e^{i\omega_0 t} + CCT]^3 \\
 &\quad + \epsilon c_{21} S_1 M (a u e^{i\omega_0 t} + CCT)^3 \\
 &\quad + \epsilon c_{22} S_1 (a u e^{i\omega_0 t} + CCT)^2 (a i \omega_0 e^{i\omega_0 t} + CCT) \\
 &\quad + \epsilon c_{23} S_2 (a u e^{i\omega_0 t} + CCT)^2 (a u i \omega_0 e^{i\omega_0 t} + CCT) \\
 &\quad + \epsilon \chi S_1 (a \tilde{w} e^{i\omega_0 t} + CCT)^3 \\
 &\quad + \epsilon \chi 3 S_2 (a \tilde{w} e^{i\omega_0 t} + CCT)^2 (a u i \omega_0 e^{i\omega_0 t} + CCT) \\
 &\quad + \epsilon \chi 3 S_3 (a \tilde{w} e^{i\omega_0 t} + CCT) (a u i \omega_0 e^{i\omega_0 t} + CCT)^2 \\
 &\quad + \epsilon \chi S_4 (a u i \omega_0 e^{i\omega_0 t} + CCT)^3
 \end{aligned} \tag{104}$$

where, CCT indicates complex conjugate terms and

$$\tilde{w} = M u + i \omega_0 \tag{105}$$

Equations 103 and 104 may be written as

$$\begin{aligned}
 g_{53} &= a^3 g_{53}^{(3)} e^{i3\omega_0 t} + a^2 a^* g_{53}^{(1)} e^{i\omega_0 t} + CCT \\
 g_{u3} &= a^3 g_{u3}^{(3)} e^{i3\omega_0 t} + a^2 a^* g_{u3}^{(1)} e^{i\omega_0 t} + CCT
 \end{aligned} \tag{106}$$

where

$$\begin{aligned}
 g_{53}^{(1)} &= \frac{1}{6} k_5 \delta 3 u^2 u^* - k'_5 3 (1 + \delta u)^2 (1 + \delta u^*) \\
 &\quad + \epsilon [c_{11} S_0 M 3 u^2 u^* + c_{12} S_0 (2 u u^* - u^2) i \omega_0 + c_{13} S_1 u^2 u^* i \omega_0] \\
 &\quad + \epsilon \chi [S_0 3 \tilde{w}^2 \tilde{w}^* + 3 S_1 (2 \tilde{w} \tilde{w}^* u - \tilde{w}^2 u^*) i \omega_0 \\
 &\quad + 3 S_2 (\tilde{w} 2 u u^* - \tilde{w}^* u^2) \omega_0^2 + S_3 3 u^2 u^* i \omega_0^3]
 \end{aligned} \tag{107}$$

and

$$\begin{aligned}
 g_{\alpha 3}^{(1)} = & \frac{1}{2} k_3 \delta (2 u u^* + u^2) + (\frac{2}{3} k_3 \delta^2 - k'_\alpha) 3 u^2 u^* \\
 & - k'_3 \delta 3 (1 + \delta u)^2 (1 + \delta u^*) \\
 & + 5 [c_{21} S_1 M 3 u^2 u^* + c_{22} S_1 (2 u u^* - u^2) i \omega_0 + c_{23} S_2 u^2 u^* i \omega_0] \\
 & + 5 \chi [S_1 3 \tilde{w}^2 \tilde{w}^* + 3 S_2 (2 \tilde{w} \tilde{w}^* u - \tilde{w}^2 u^*) i \omega_0 \\
 & + 3 S_3 (\tilde{w} 2 u u^* - \tilde{w} u^2) \omega_0^2 + S_0 3 u^2 u^* i \omega_0^3] \quad (108)
 \end{aligned}$$

Similar expression for  $g_{\beta 3}^{(2)}$  and  $g_{\alpha 3}^{(3)}$  are not explicitly given here because they are not needed in the current analysis.

The condition to avoid secular terms, Eq. 29, yields Eq. 31 where now (Eqs. 7B, 96 and 98)

$$\begin{aligned}
 \beta = -\lambda_2 \underline{v}^T \underline{A}_2 \underline{u} &= \frac{\lambda_2}{\alpha_0} \begin{Bmatrix} v_1 \\ v_2 \\ 1 \\ v v \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & S_0 & 0 & 0 \\ 0 & -S_1/v & 0 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ u \\ i \omega_0 \\ i \omega_0 u \end{Bmatrix} \\
 &= \frac{\lambda_2}{\alpha_0} [1 \quad v v] \begin{bmatrix} 0 & S_0 \\ 0 & -S_1/v \end{bmatrix} \begin{Bmatrix} 1 \\ u \end{Bmatrix} = \lambda_2 (S_0 - v S_1) u / \alpha_0 \quad (109)
 \end{aligned}$$

and

$$\gamma = \underline{v}^T \underline{f}_3^{(1)} = \frac{1}{\alpha_0} \begin{Bmatrix} v_1 \\ v_2 \\ 1 \\ v v \end{Bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ g_{\beta 3}^{(1)} \\ g_{\alpha 3}^{(1)}/v \end{Bmatrix} = (g_{\beta 3}^{(1)} + v g_{\alpha 3}^{(1)}) / \alpha_0 \quad (110)$$

The limit cycle amplitude is then given by (see Eqs. 47 and 50)

$$|a|_{LC} = \left| \frac{\text{Real}[(S_0 u - v S_1 u) / \alpha_0]}{\text{Real}[(g_{\beta 3}^{(1)} + v g_{\alpha 3}^{(1)}) / \alpha_0]} \right|^{1/2} \quad (111)$$

The numerical results are discussed in Section V.

#### 4.4 Flutter Taming

In order to implement the flutter-taming formulation, the system considered above must be modified. Three modifications are discussed here (the details are given in Appendix B). The simplest one consists in changing the nonlinear-stiffness constant  $k'_\alpha$  (this case "simulates" a modification of the structural design of the aircraft to insure benign flutter). Another simple approach consists in adding a nonlinear damper (this case "simulates" a modification of existing aircraft). These two approaches may be studied using the same equations used above where now  $g_\alpha$  is given by Eq. 81, or

$$\hat{g}_\alpha = g_\alpha - \kappa' \dot{\alpha}^3 \quad (112)$$

This implies that  $\gamma^*$  is given by Eq. 70 where  $\kappa'' = 0$  and (see Eq. 72)

$$\gamma^* = -v i \omega_0^3 3 u^2 u^* \quad (113)$$

In a more sophisticated approach one can make use of a control surface and nonlinear active control. The presence of the control surface introduces forces and moments (where  $\theta$  is the deflection of the control surface). The equation of the actuator dynamic is assumed to have an input proportional for instance to  $\ddot{z}^3$  (e.g., the doubly-integrated signal from an accelerometer located in C). In this case the governing equations are given by Eq. B.69 which may be recast in the form of Eq. 54 with  $\kappa' = 0$  (see however Eq. 112) and

$$\underline{y} = \begin{Bmatrix} \theta \\ \dot{\theta} \end{Bmatrix} \quad (114)$$

$$\underline{B}(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -\pi S_\xi^{(n)} \\ 0 & \pi S_\alpha^{(n)}/v \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -S_\xi^{(0)} & 0 \\ S_\alpha^{(0)}/v & 0 \end{bmatrix} \quad (115)$$

$$\underline{D} = \begin{bmatrix} 0 & 1 \\ -k/J & -r/J \end{bmatrix} \quad (116)$$

$$\underline{z} = \begin{Bmatrix} 0 \\ 1/J \end{Bmatrix} \quad (117)$$

$$\underline{c}^T = [1 \ 0 \ 0 \ 0] \quad (118)$$

This yields (see Eq. 65)

$$\begin{aligned} \underline{\check{v}} &= [i\omega_0 \underline{I} - \underline{D}^T]^{-1} \underline{B}_0^T \underline{v} \\ &= \begin{bmatrix} i\omega_0 & k/J \\ -1 & i\omega_0 + r/J \end{bmatrix}^{-1} \begin{Bmatrix} \lambda (-S_s^{(0)} + v S_\alpha^{(0)}) \\ \epsilon (-S_s^{(0)} + v S_\alpha^{(0)}) \end{Bmatrix} = \begin{Bmatrix} \check{v}_1 \\ J \check{v} \end{Bmatrix} \end{aligned} \quad (119)$$

or

$$\check{v} = [i\omega_0 \epsilon (-S_s^{(0)} + v S_\alpha^{(0)}) + \lambda (-S_s^{(0)} + v S_\alpha^{(0)})] / (\omega_0^2 J + i\omega_0 r + k) \quad (120)$$

and a similar expression for  $\check{v}_1$  which is inessential here.

The expression for  $\check{\gamma}$  is given by Eq. 70 where  $\gamma'$  is given by Eq. 113 (with  $\kappa'$  if the damper is not added) and  $\check{\gamma}''$  given by Eq. 73, or

$$\check{\gamma}'' = 3\mu^2 \mu^* \underline{\check{v}}^T \underline{z} = 3\mu^2 \mu^* \check{v} \quad (121)$$

with  $\check{v}$  given by Eq. 120 and  $\mu$  given by Eq. 69 or

$$\mu = \underline{c}^T \underline{u} = 1 \quad (122)$$

combining Eqs. 120, 121 and 122 one finally obtains

$$\check{\gamma}'' = 3 [i\omega_0 \epsilon (-S_s^{(0)} + v S_\alpha^{(0)}) + \lambda (-S_s^{(0)} + v S_\alpha^{(0)})] / (\omega_0^2 J + i\omega_0 r + k) \quad (123)$$

In the design of the actuator, the parameters  $J$ ,  $r$  and  $k$  may be used to insure that  $\check{\gamma}'' \neq 0$  (or even  $\check{\gamma}'' = \text{optimal}$ )

Numerical results are considered in Section V.

## SECTION V

### NUMERICAL RESULTS

#### 5.1 Introduction

In order to assess the feasibility of flutter taming (e.g., whether the modifications may be implemented with physically acceptable values for the constants  $\kappa'$  and  $\kappa''$ ), specific numerical results are presented here. All the results are obtained using a configuration similar to that of Ref. 15 and 18. In particular the following values are used for the constants  $\xi_0$ ,  $\delta$ ,  $\nu$ ,  $k_5$  and  $k'_5$

$$\begin{aligned}\xi_0 &= -.2 \\ \delta &= -\xi_0 = .2 \\ \nu &= .21 \\ k_5 &= 0. \\ k'_5 &= 0. \dots \dots \dots (124)\end{aligned}$$

#### 5.2 Flutter Analysis of Original System

Consider first the linear system according to the analysis of Section IV the flutter frequency is given by Eq. 94 or, using the above values for  $k_5$ ,  $\nu$  and  $\xi_0$ , one obtains

$$\omega_0^2 = \frac{1}{.21 + 1/3 + .04} k_\alpha = \frac{12}{7} k_\alpha = .36 \Omega_\alpha^2 \quad (125)$$

where (as in Ref. 15)

$$\Omega_\alpha = \sqrt{k_\alpha / \nu} \quad (126)$$

The flutter boundary is given by Eq. 95 and is plotted as a function of  $\sqrt{k_\alpha}$  in Figure 7 (for  $1/\sigma = 4, 8, 12$  and  $16$ ). The corresponding Mach number,  $M_0 = \lambda_0/\sigma$  (see Eq. 80), is plotted in Figure 8 as a function of  $\sqrt{k_\alpha}$  (for the same values of  $\sigma$ ). The above results coincide, as expected, with those of Ref. 15.

In the nonlinear analysis two formulations have been presented. The two differ only for the direction of the downwash used in the piston theory. In the first one, the



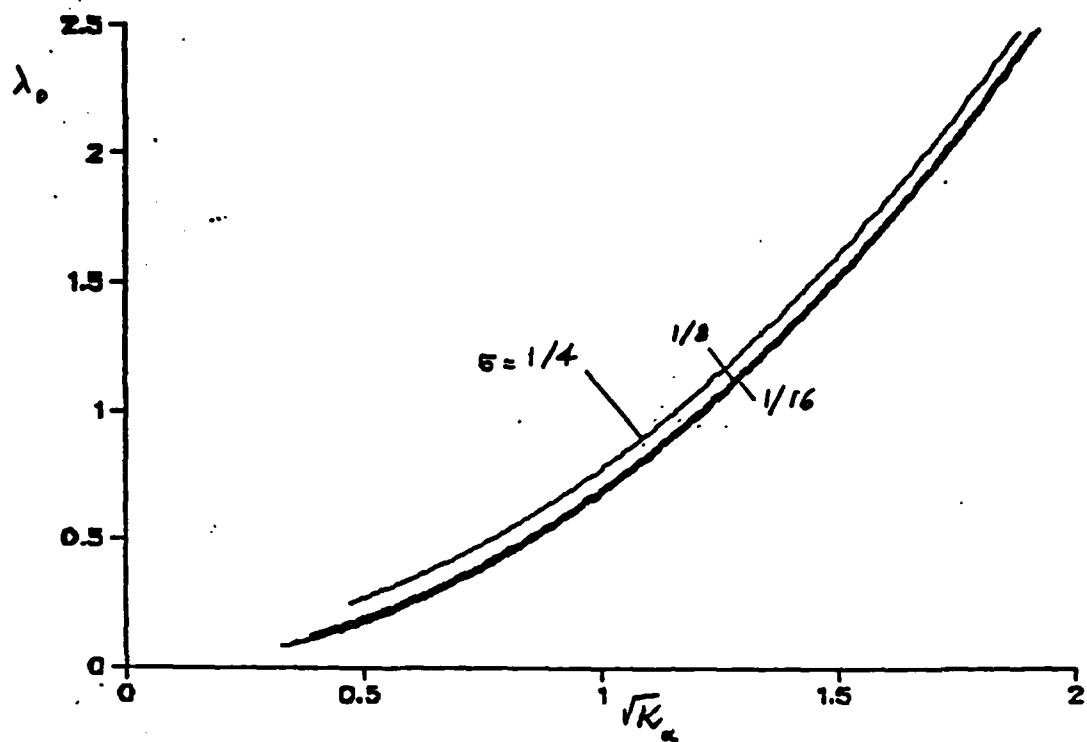


Figure 7

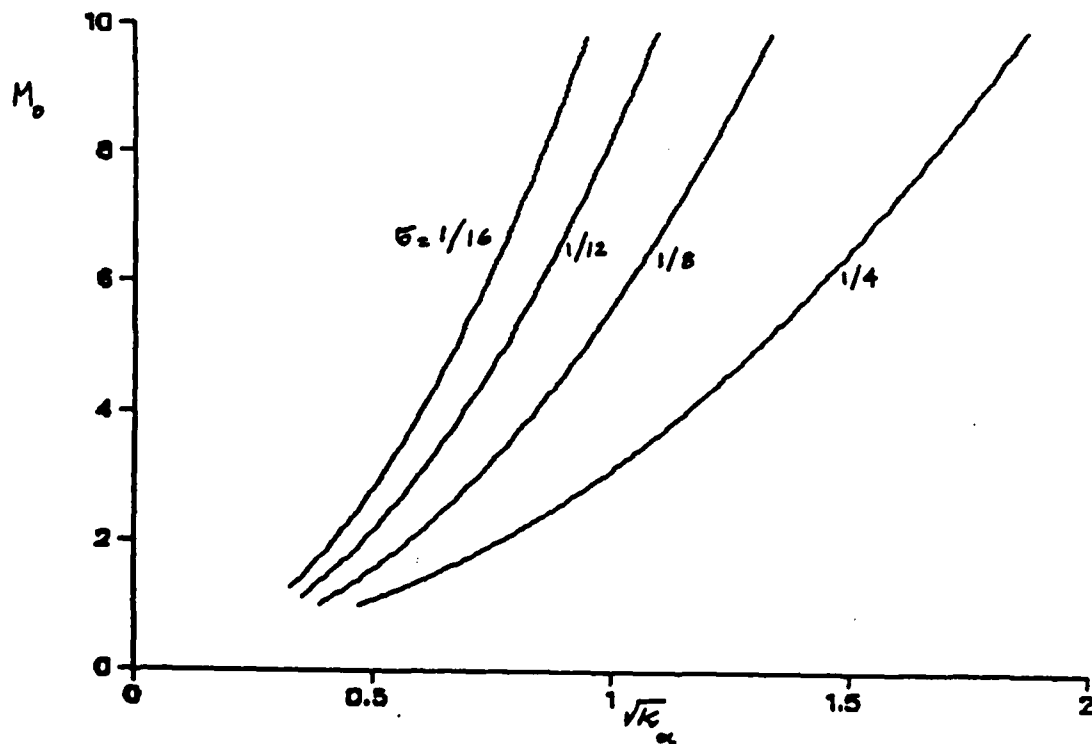


Figure 8

downwash is in the vertical direction (akin to the formulation of Ref. 15). I will refer to this formulation as type V (for vertical). In the second one, the downwash is in the local direction of the normal to the airfoil surface: I will refer to this formulation as type N (for normal). Equation 85 is used for type V, whereas Eq. 84 is used for type N. The value of the limit-cycle amplitude (as given by Eq. 111), are plotted, in Figures 9-12 (for  $1/\epsilon = 4, 8, 12$  and 16 respectively), as a function of  $\sqrt{k_\alpha}$  for several values of  $k_\alpha$  (values of  $k_\alpha$  in parenthesis indicate an unstable limit cycle). The results obtained using the first formulation (type V), shown on the upper side of the figures coincide, as expected, with those of Ref. 15. Those obtained using the second formulation (type N) are shown on the bottom of the figures and indicate that the difference between the two approaches, while not dramatic is still significant. From an physically intuitive point of view the second approach is obviously more satisfactory than the first one. However I am not aware of any rigorous proof, that indeed the second approach is the correct one. Therefore in the following all the figures include both, type V (top) and type N (bottom) results.

### 5.3 Taming by Stiffening

Figures 9-12 indicate that as  $k'_\alpha$  increases, the limit cycles tend to become stable. This suggested the first, simple approach to the flutter taming: modify the value of the nonlinear stiffness constant  $k'_\alpha$  in order to achieve flutter taming. The results are shown in Figures 13-16 (for  $1/\epsilon = 4, 8, 12$  and 16 respectively) where the limit cycle amplitude is plotted versus  $k'_\alpha/k_\alpha$  for several values of  $\sqrt{k_\alpha}$  (values of  $\sqrt{k_\alpha}$  in parenthesis indicate an unstable cycle): as mentioned above the downwash expression of Ref. 15 (type V, Eq. 85) is used for the top graph, whereas the formulation of type N (Eq. 84) is used for the bottom graph. These figures are a different way of presenting the same data shown in Figures 9-12. It is apparent that positive values of  $k'_\alpha$  tend to stabilize the limit cycle; for instance, on Figure 13 ( $\epsilon = 1/4$ ), for values of  $k'_\alpha/k_\alpha = 3$  and for all the curves except for  $\sqrt{k_\alpha} = 1.5$ , the limit cycle amplitudes are relatively small. Note that if  $|\lambda|_{L.C.} \approx 1$ , then the value of the solution will be of the order  $(\lambda - \lambda_0)^{1/2}$ . This means, for instance, that if  $\lambda$  exceeds  $\lambda_0$  by 1% (i.e., if the flight speed exceeds the flutter speed by 1%), then the vertical displacement will be equal to .1b (5% of the chord). While this value is relatively small, it should be emphasized that further reduction in the limit cycle amplitude require very high values of  $k'_\alpha$ , hard to obtain in an actual aircraft

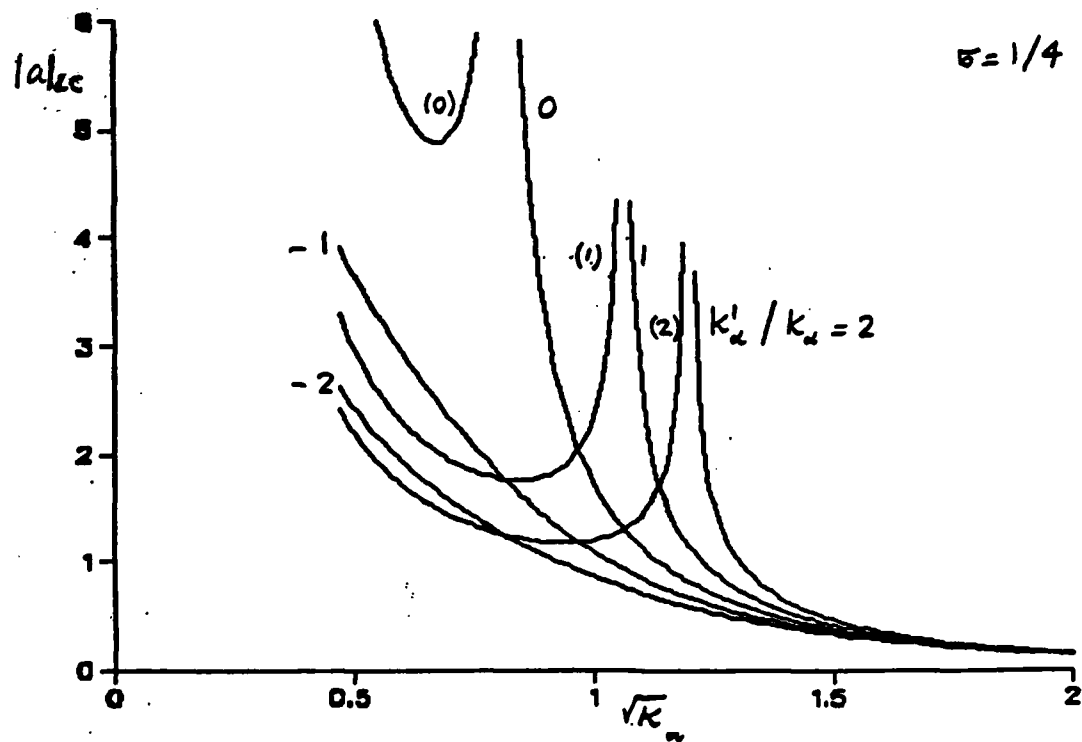
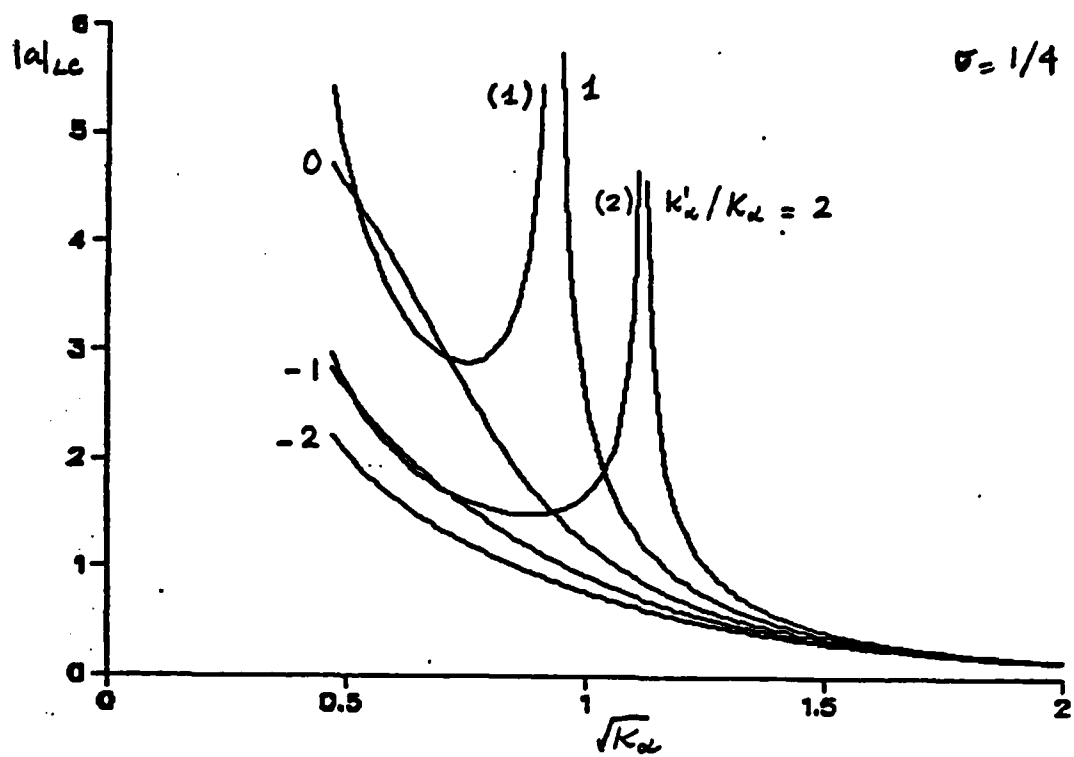


Figure 9

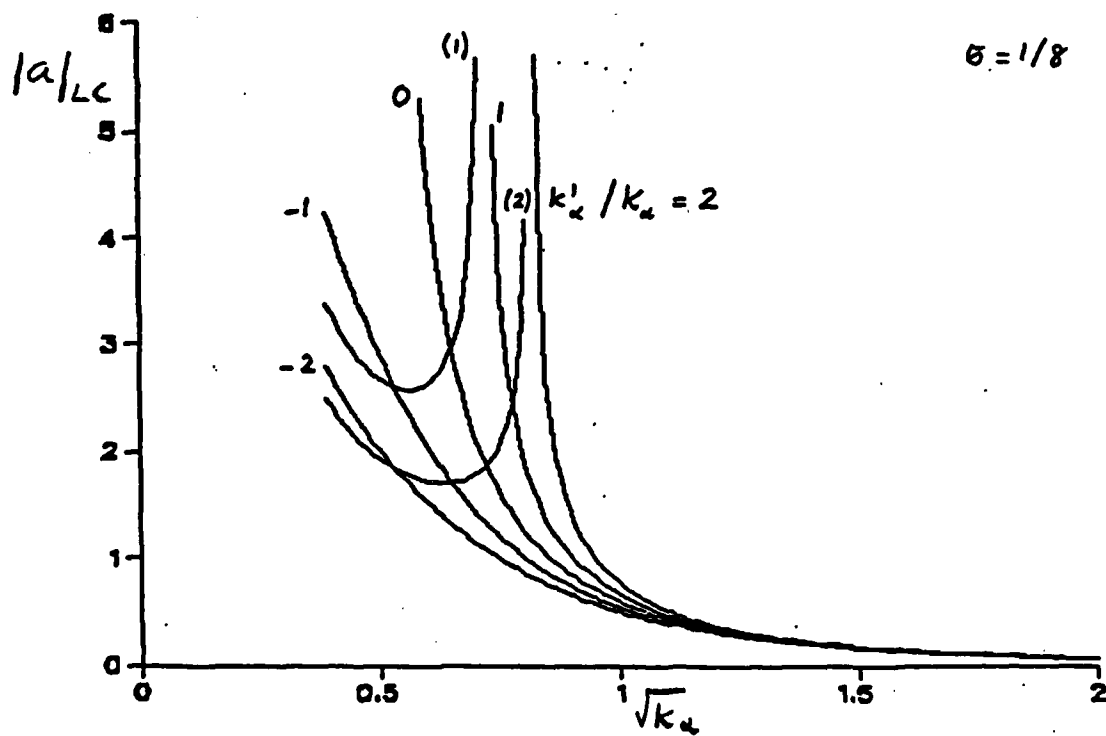
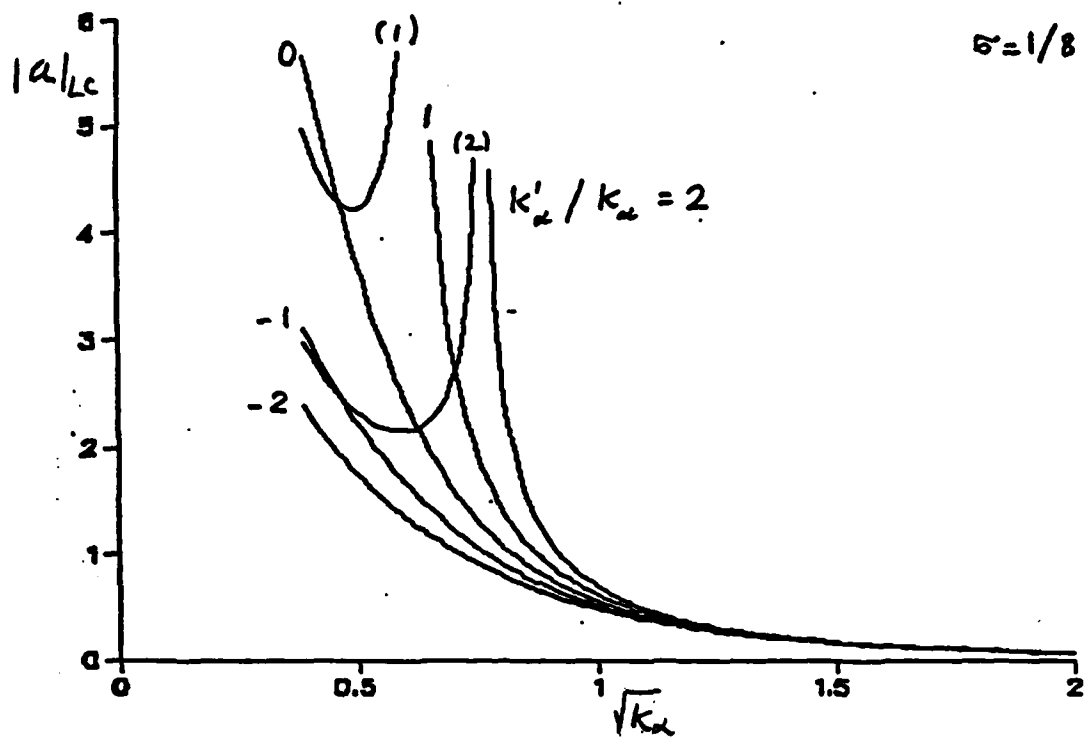


Figure 10

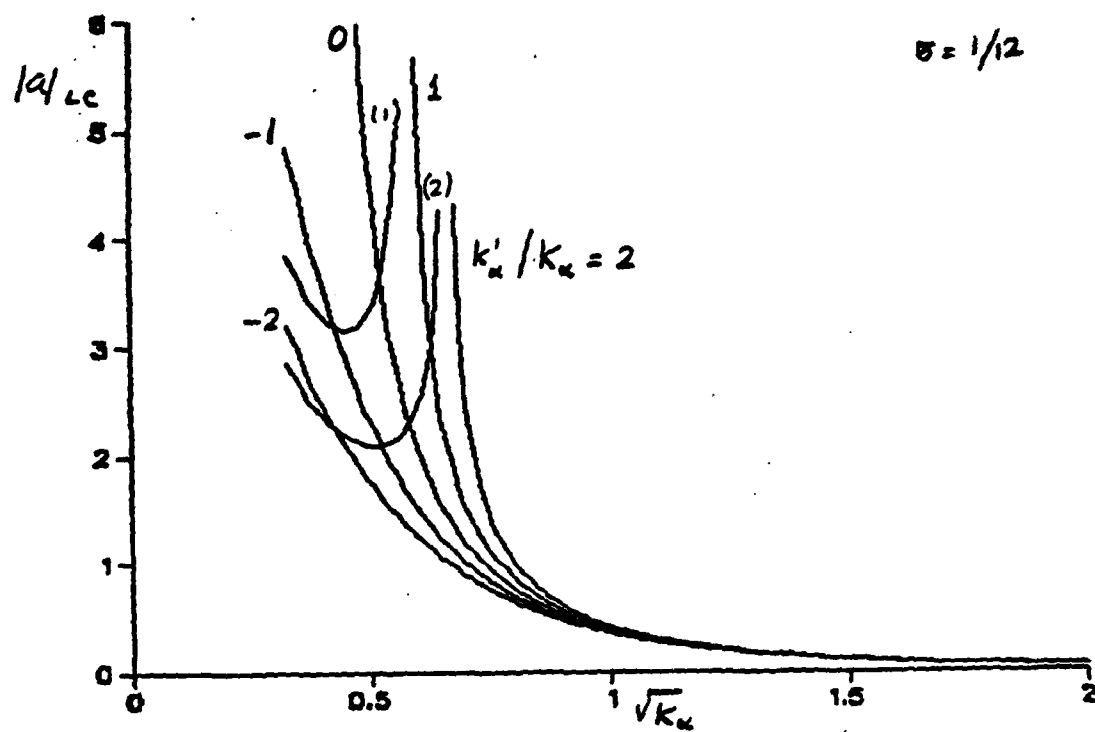
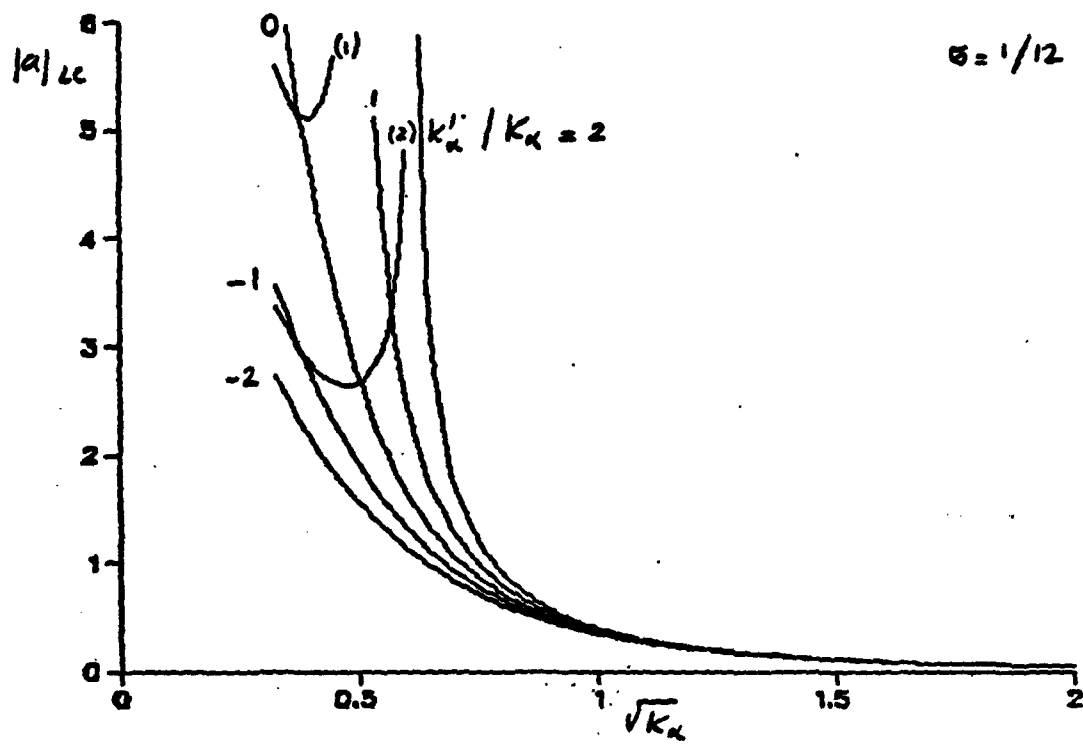


Figure 11

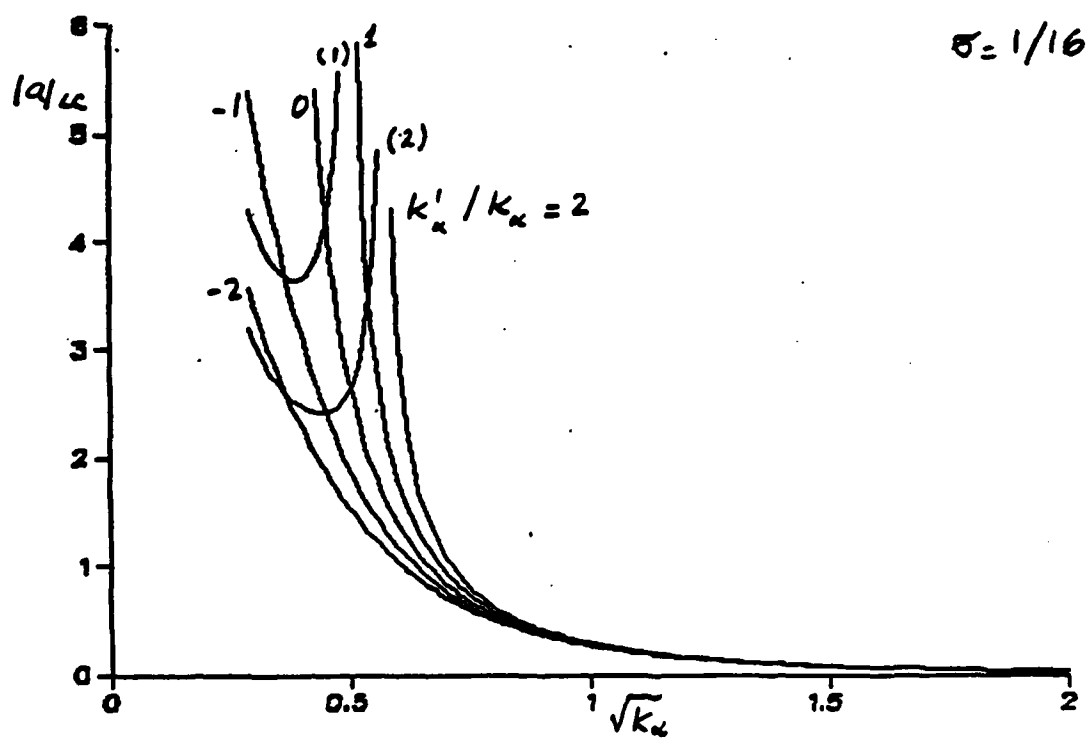
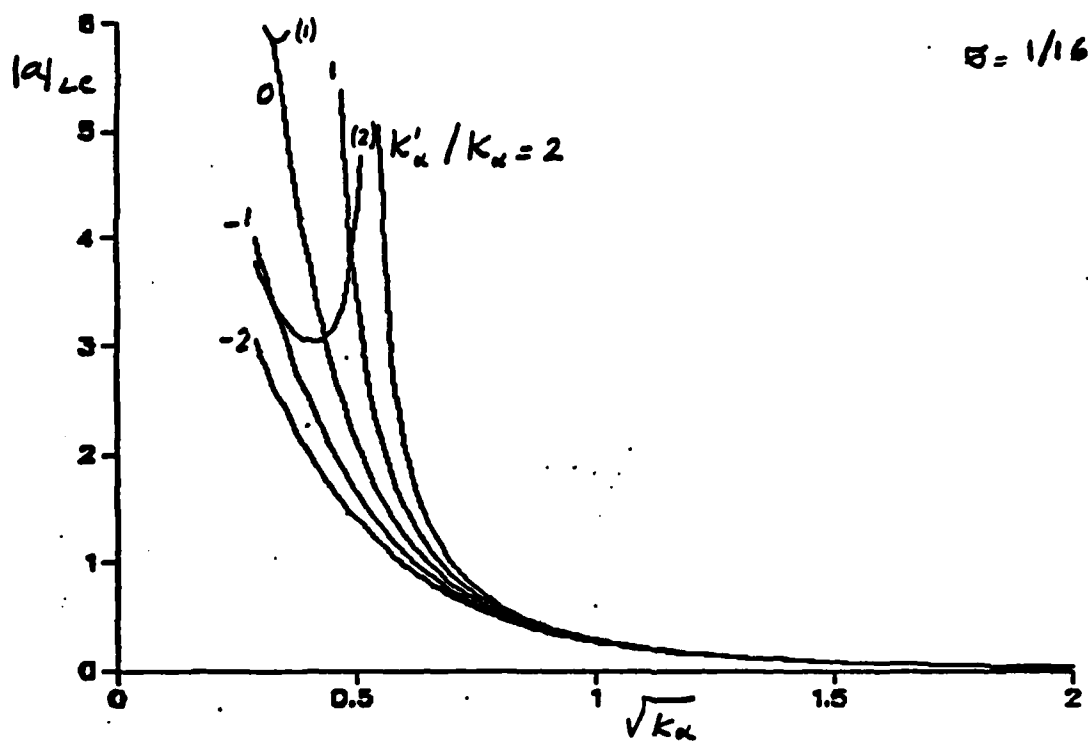


Figure 12

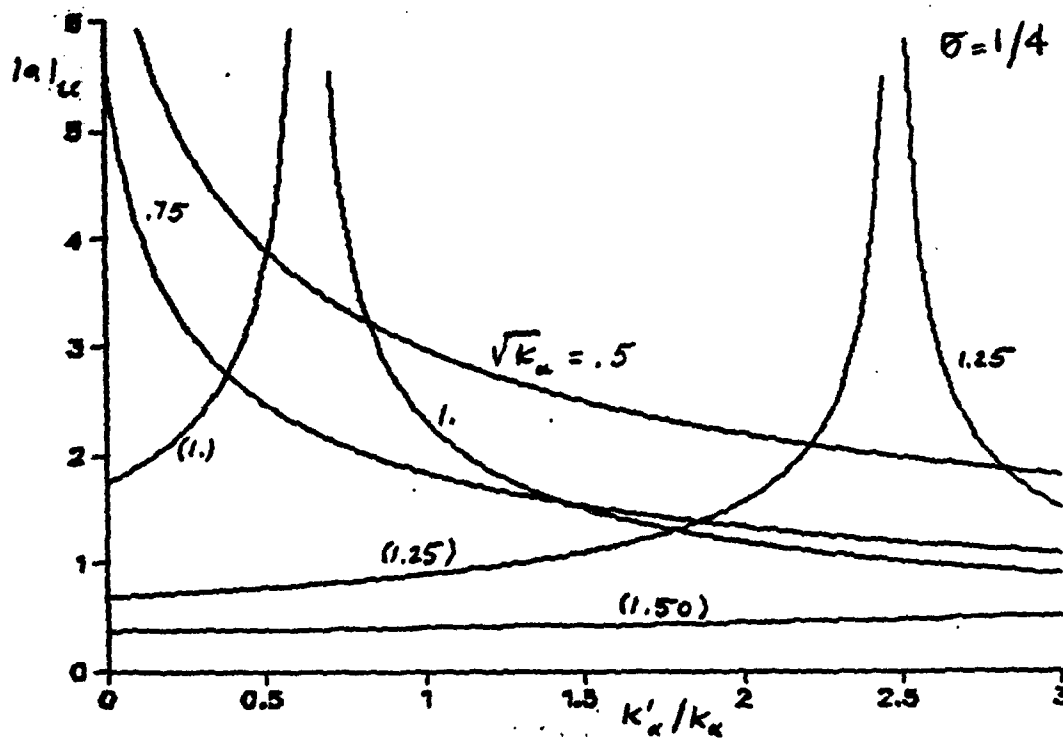
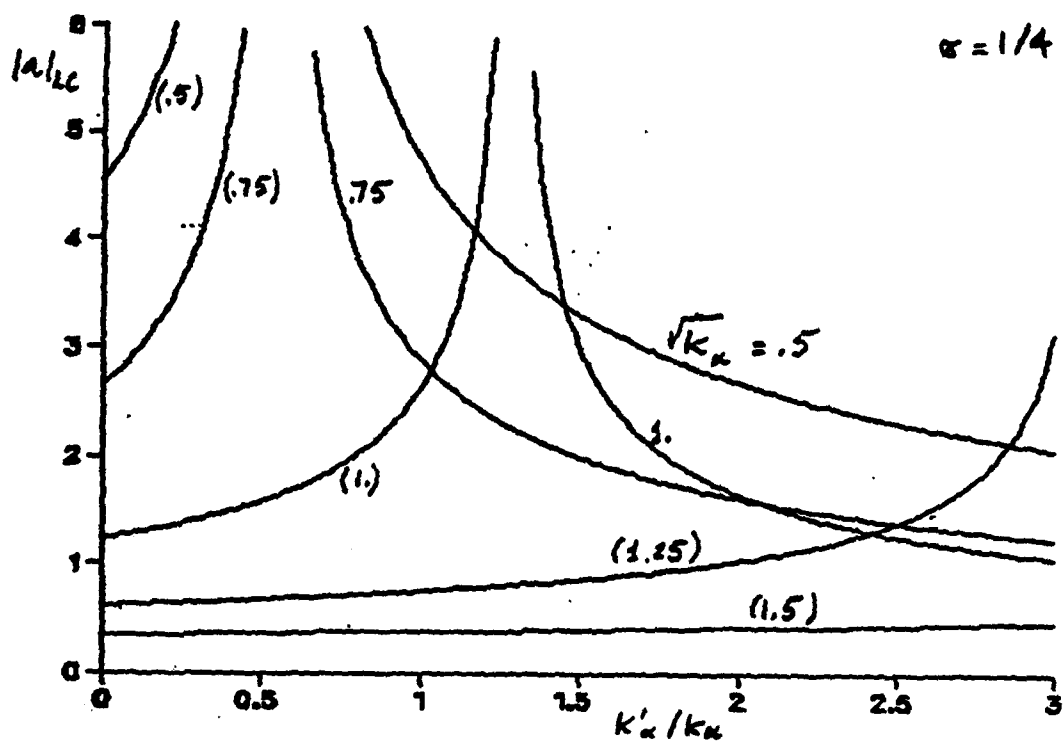


Figure 13

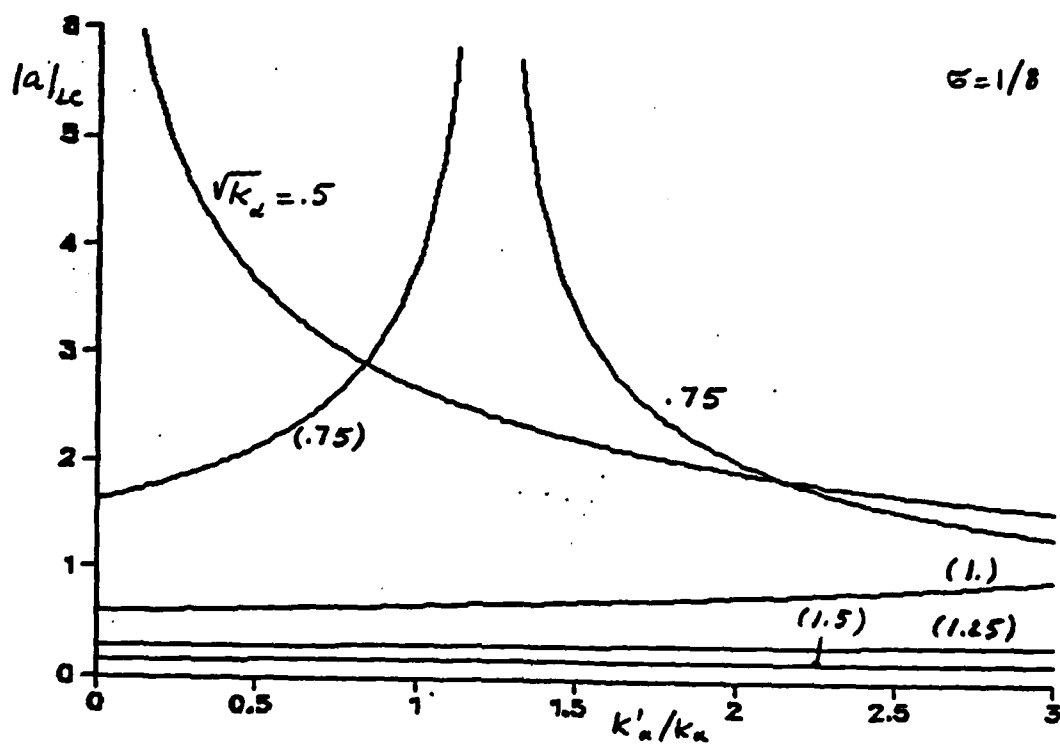
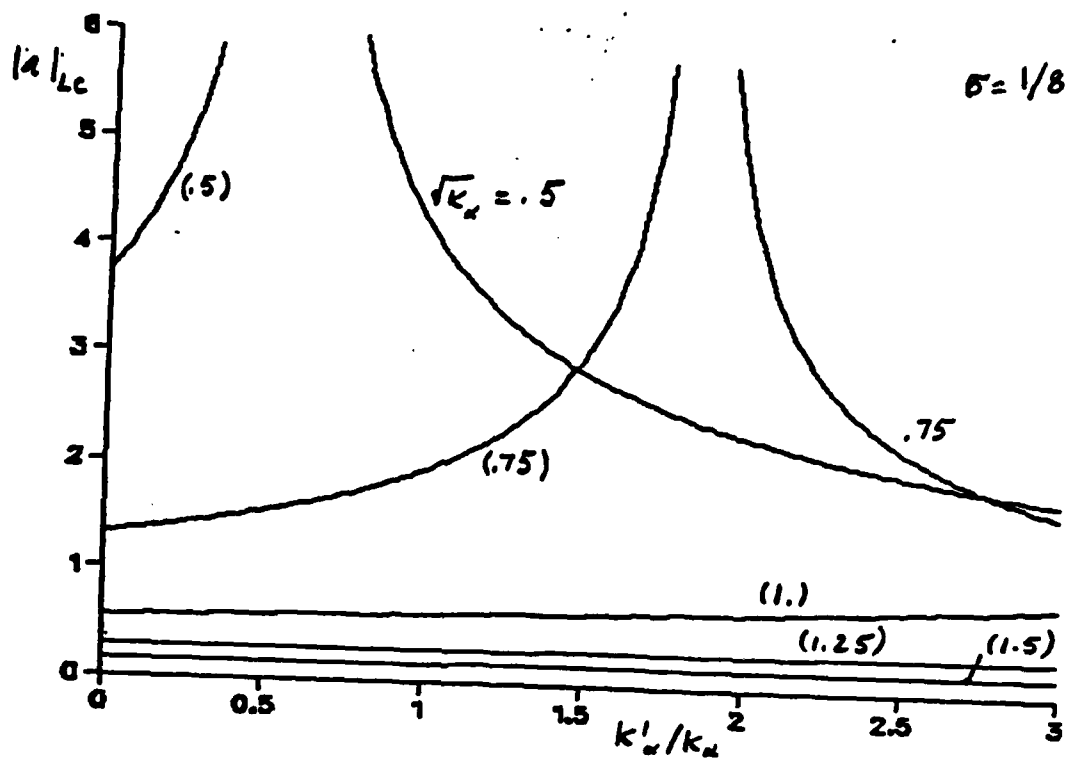


Figure 14



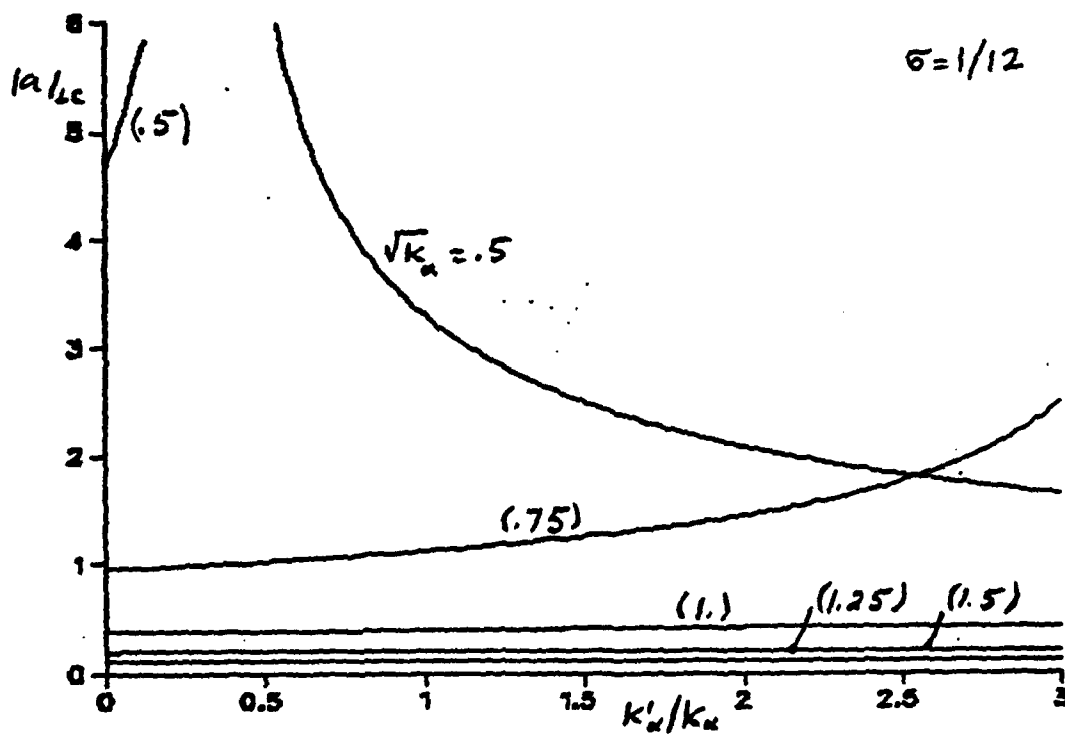
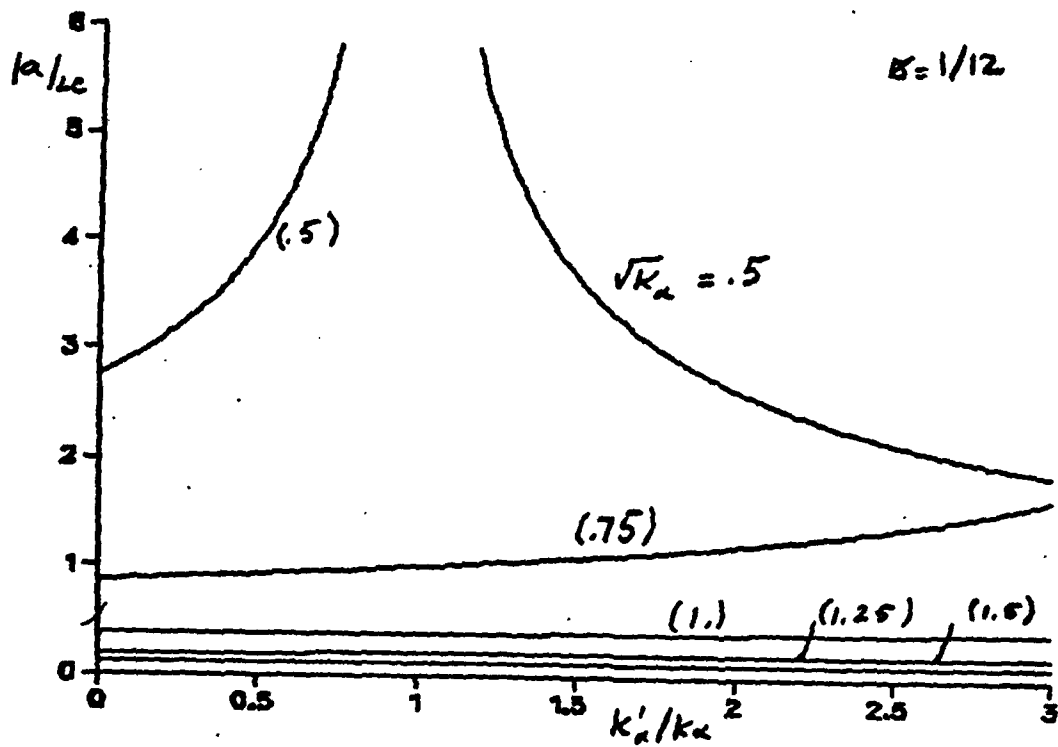


Figure 13

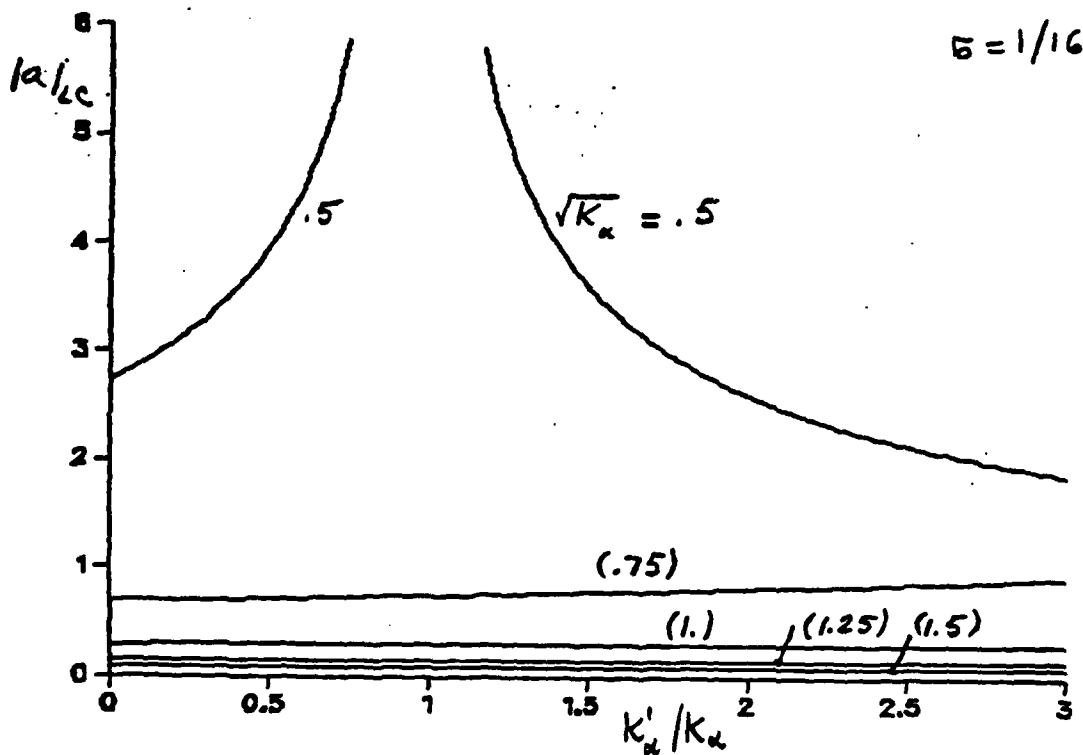
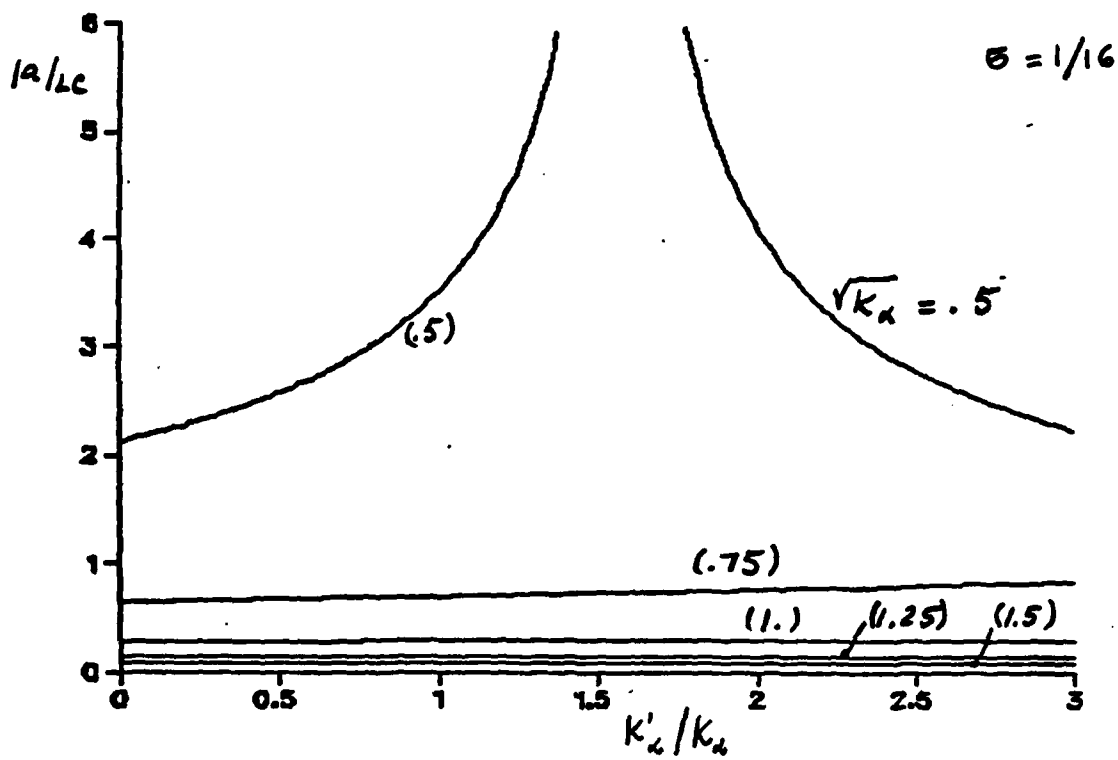


Figure 16

design. In addition for  $\sigma = 1/8$  (Figure 14) the curves for  $\sqrt{k_c} = 1, 1.25$ , and  $1.5$  are not stabilized even for  $k_c/k_c = 3$ . For  $\sigma = 1/12$  and  $1/16$  (Figures 15 and 16) only the curve for  $\sqrt{k_c} = .5$  is stabilized.

#### 5.4 Taming by Damping

Equally simple (at least from a mathematical modeling viewpoint) is the taming by addition of a damper. The results (see Eq. 70 with  $\kappa''=0$  and  $\gamma'$  given by Eq. 113) are shown in Figures 17-20 (for  $1/\sigma = 4, 8, 12$  and  $16$  respectively) where the limit cycle amplitude versus  $\kappa'$  is plotted for several values of  $\sqrt{k_c}$ : values of  $\sqrt{k_c}$  in parenthesis indicate an unstable limit cycle (type V formulation, Eq. 85, is used for the top of the figure, type N formulation Eq. 84, is used for the bottom). It may be noted that positive values of the constant  $\kappa'$  (i.e., positive damping) have stabilizing effects (whereas this might not come as a surprise, this point deserves further attention and points to the need for the development of a mathematical formulation which would identify under what conditions nonlinear damping terms are stabilizing). The same comments made for the taming by stiffening: for  $\sigma = 1/4$  (Figure 17), small values of the constant  $\kappa'$  yield a stabilization and then a dramatic reduction on the limit cycle amplitude: however further reductions require large values for the constant. For  $\sigma = 1/8, 1/12, 1/16$  (Figures 18-20) similar trends are observed, but stabilization is not achieved for the higher values of  $k_c$ .

#### 5.5 Taming by Active Control

The expression for the limit cycle amplitude for a system modified with the addition of active control is derived in Sect. IV (see Eq. 70 with  $\gamma'=0$  and  $\kappa''$  given by Eq. 73). The constants  $J, r$ , and  $k$ , determine the dynamics of the actuator, Eq. B.67, and, as mentioned at the end of Sect. IV, they may be used to optimize the limit cycle value. This idea of "optimal design" was brought about by the first results obtained by choosing certain values for  $J, r$ , and  $k$ . The diagrams indicated very little influence of the feedback even for  $\kappa'' = 20$ . It turned out that those values yield a value of  $\gamma''$  very close to the imaginary axis so that  $\gamma'' \approx 0$  (if  $\gamma''=0$  the constant  $\kappa''$  has no effect on  $\gamma''$ , Eq. 70). By changing the value of the phase angle of  $-\omega^2 J + i\omega r + k$ , we found that a value for  $\gamma''$  close to the maximum is obtained for  $r/J = .35$  and  $k/J = .3$ . In view of the fact that the nondimensional moment of the inertia for the airfoil is  $\gamma = .21$ , it seems reasonable to choose  $J = .1$  and hence  $r = .035$  and  $k = .03$ . The results are plotted in Figures 21-

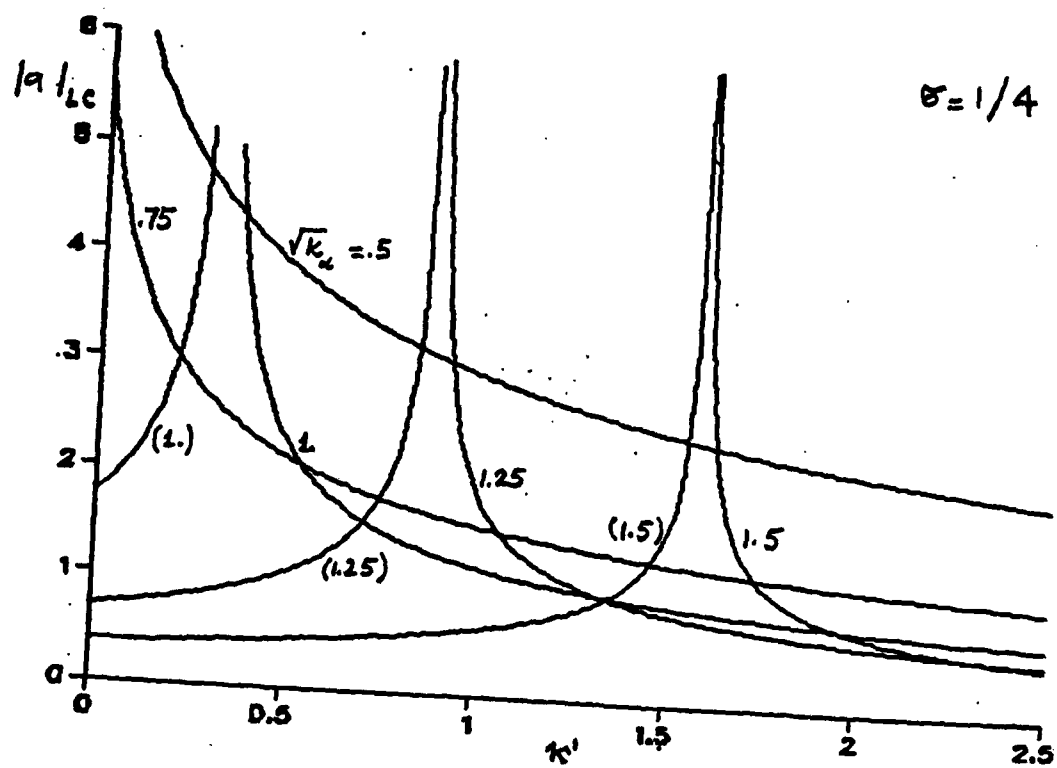
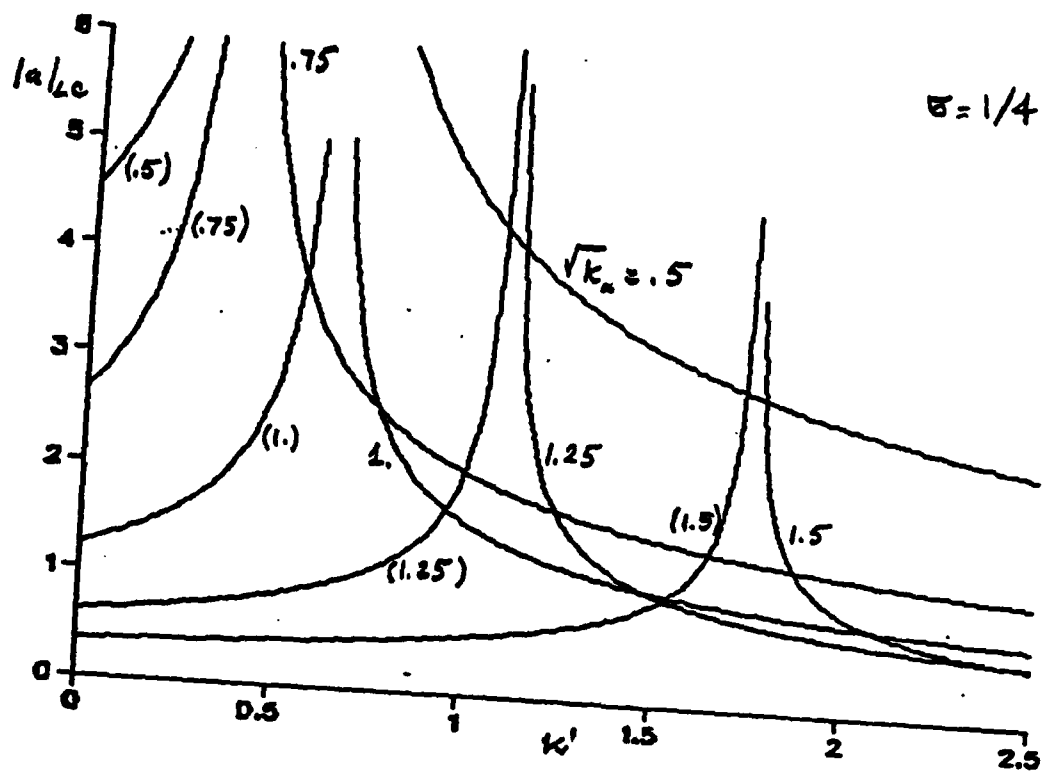


Figure 17

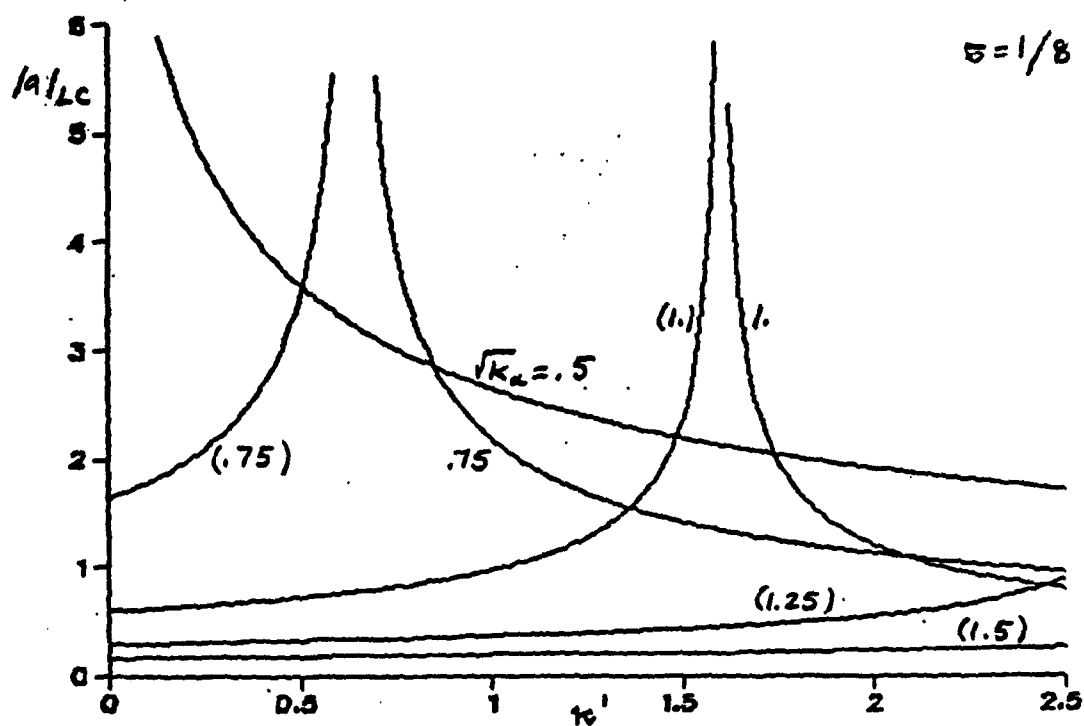
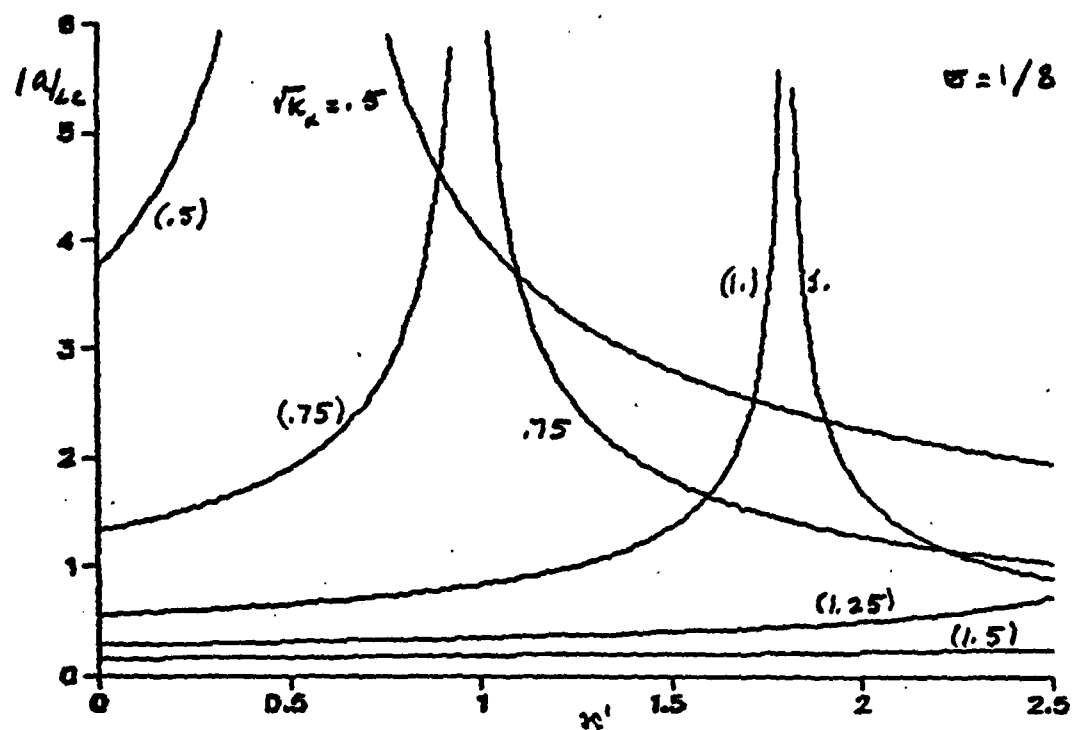


Figure 1B

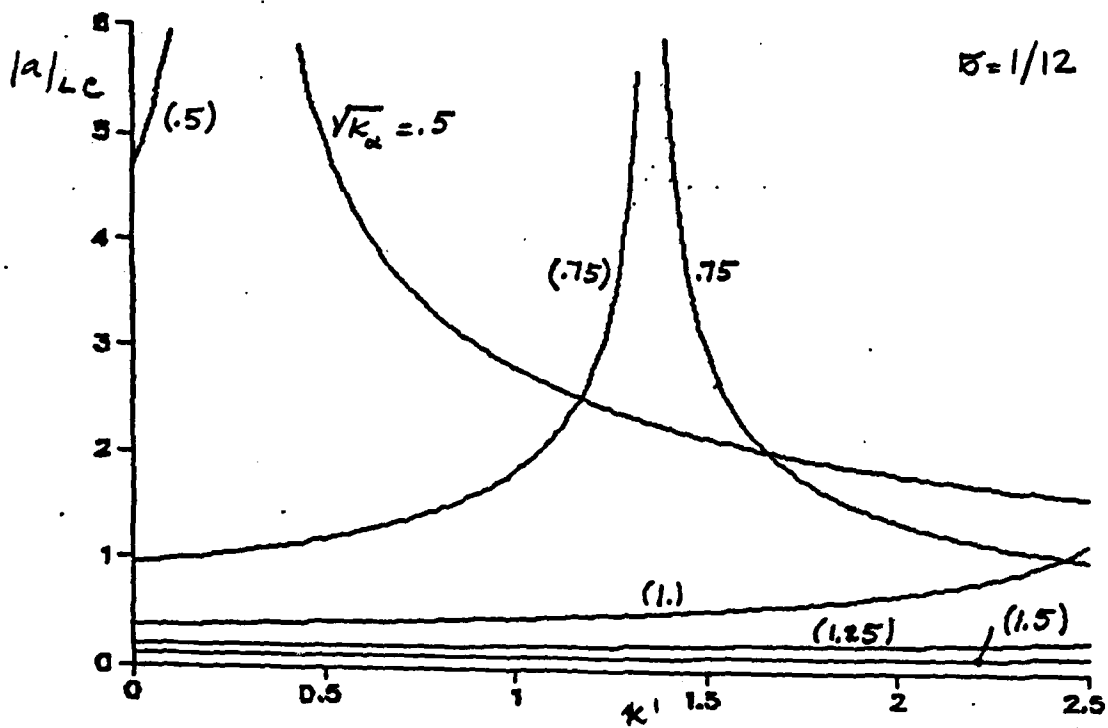
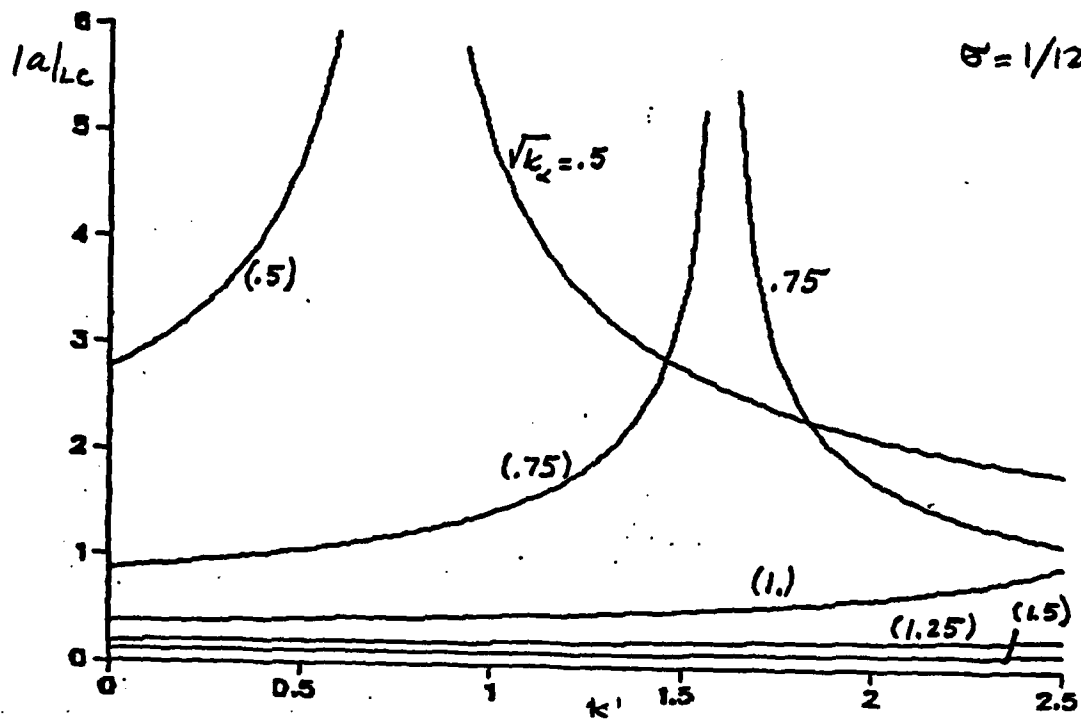


Figure 19

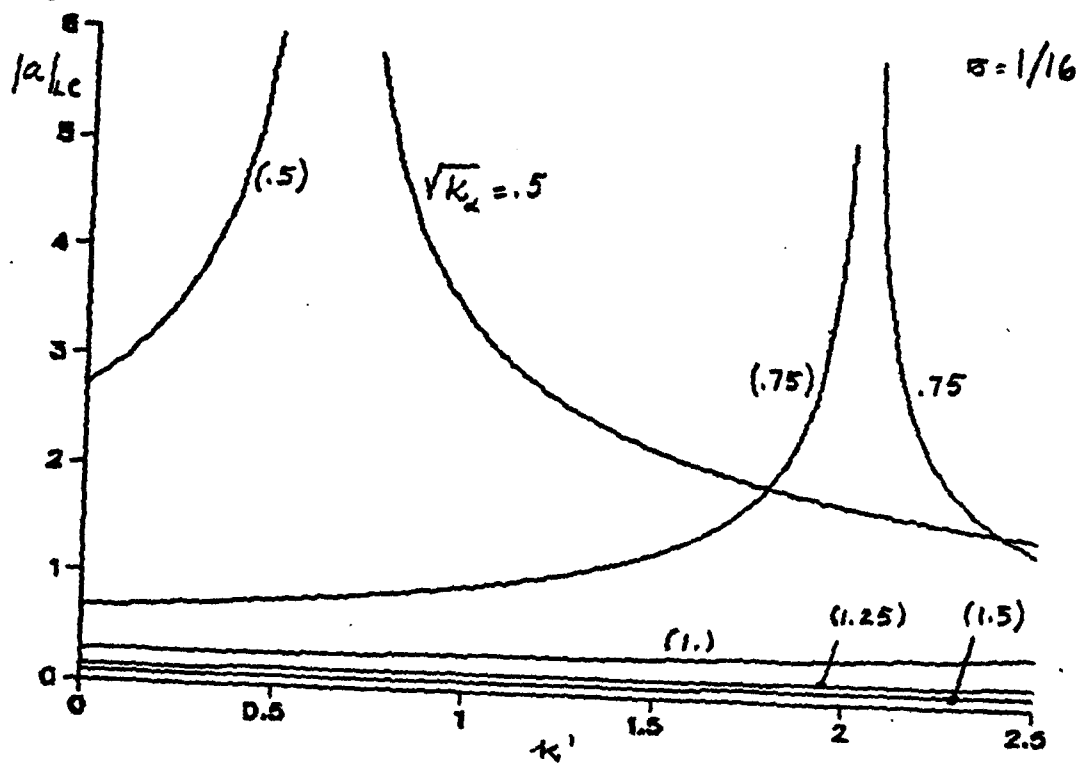
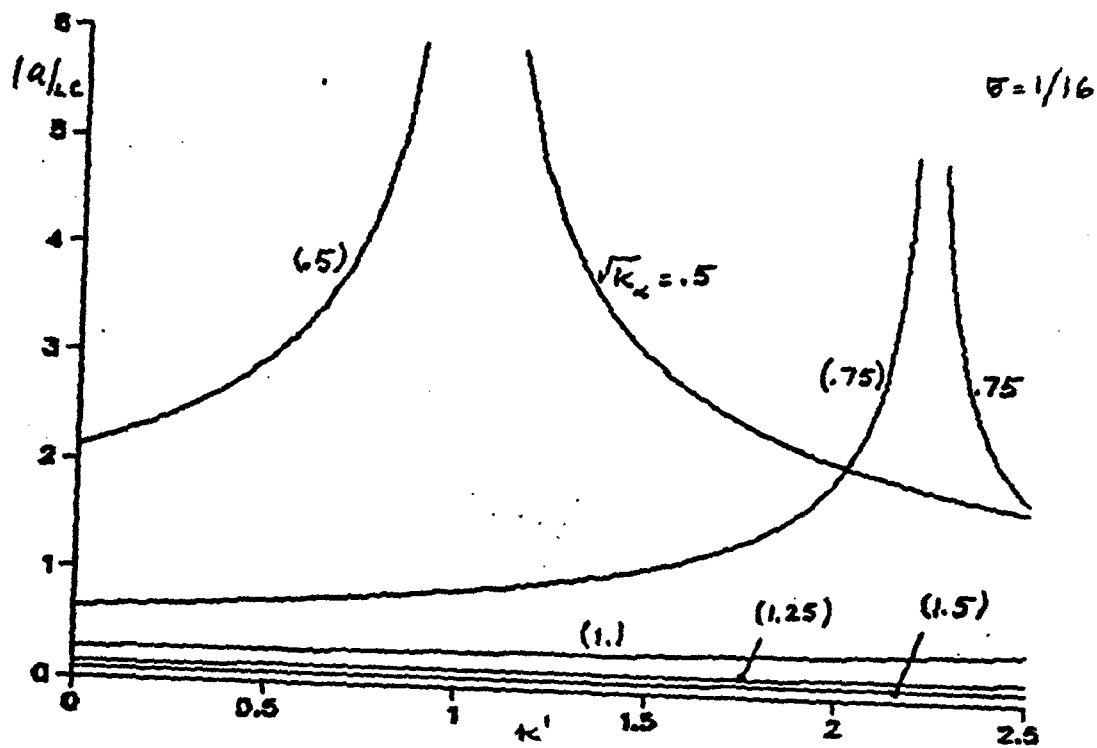


Figure 20

24 (for  $1/\epsilon=4, 8, 12, 16$  respectively), which show the limit cycle amplitude versus  $\kappa''$  for several values of  $\sqrt{K_0}$  (values of  $\sqrt{K_0}$  in parenthesis indicate unstable limit cycle). Note that this time only the curve  $\sqrt{K_0}=1.5$  (for  $\epsilon=1/8, 1/12$ , and  $1/16$ ) is not stabilized at  $\kappa''=-1$ . Otherwise, as in the case of the taming by stiffening and by damping, small values of  $\kappa''$  yield a stabilization of the unstable limit cycle and a dramatic reduction of the limit cycle amplitude. Slower reductions in the limit cycle amplitude are obtained with further increases of the gain  $-\kappa''$ : however, high values for this constant may be more easily obtainable in this case than the case of taming by stiffening or by damping. Further research is recommended on this issue.



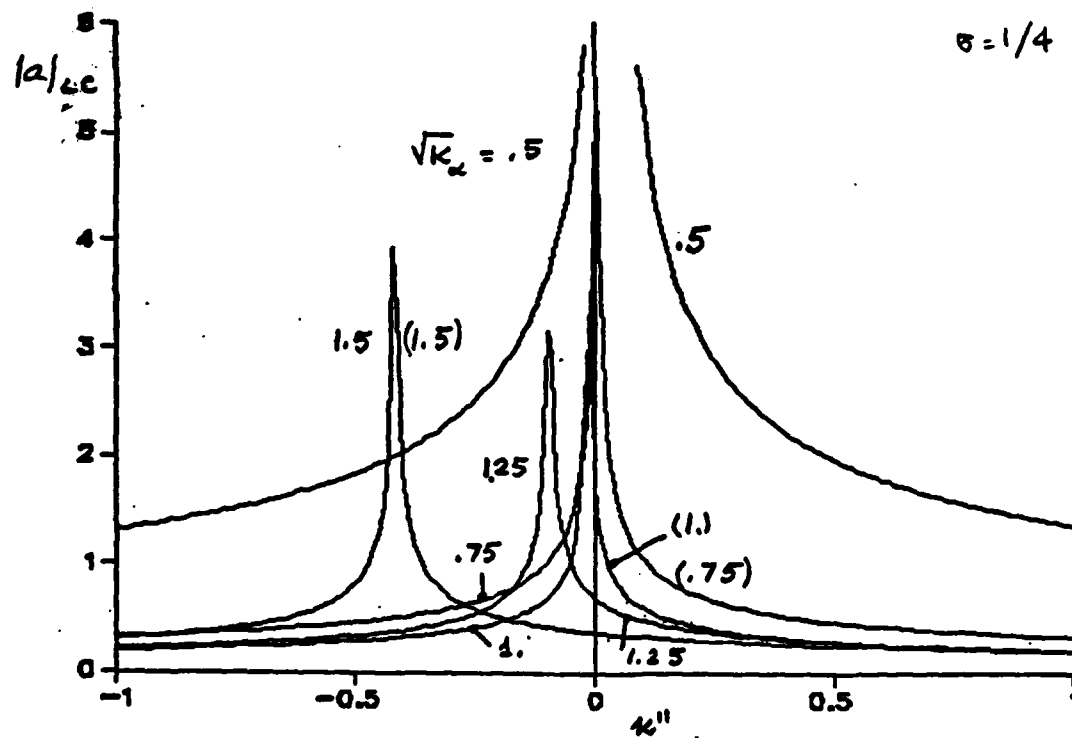
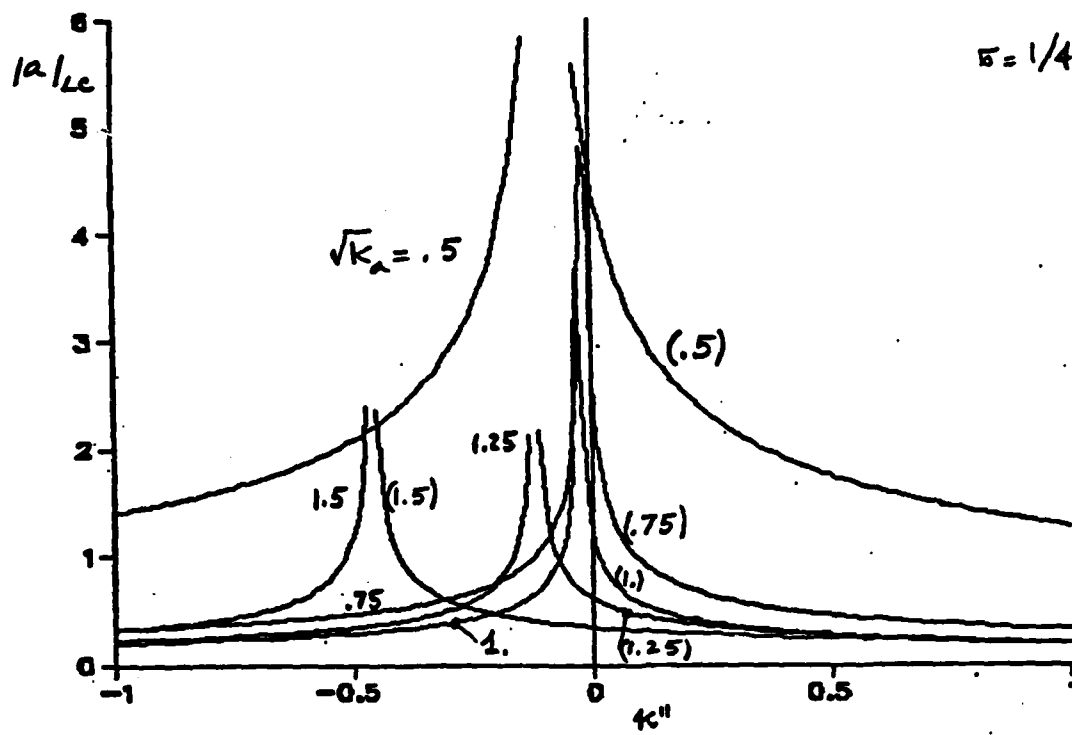


Figure 21

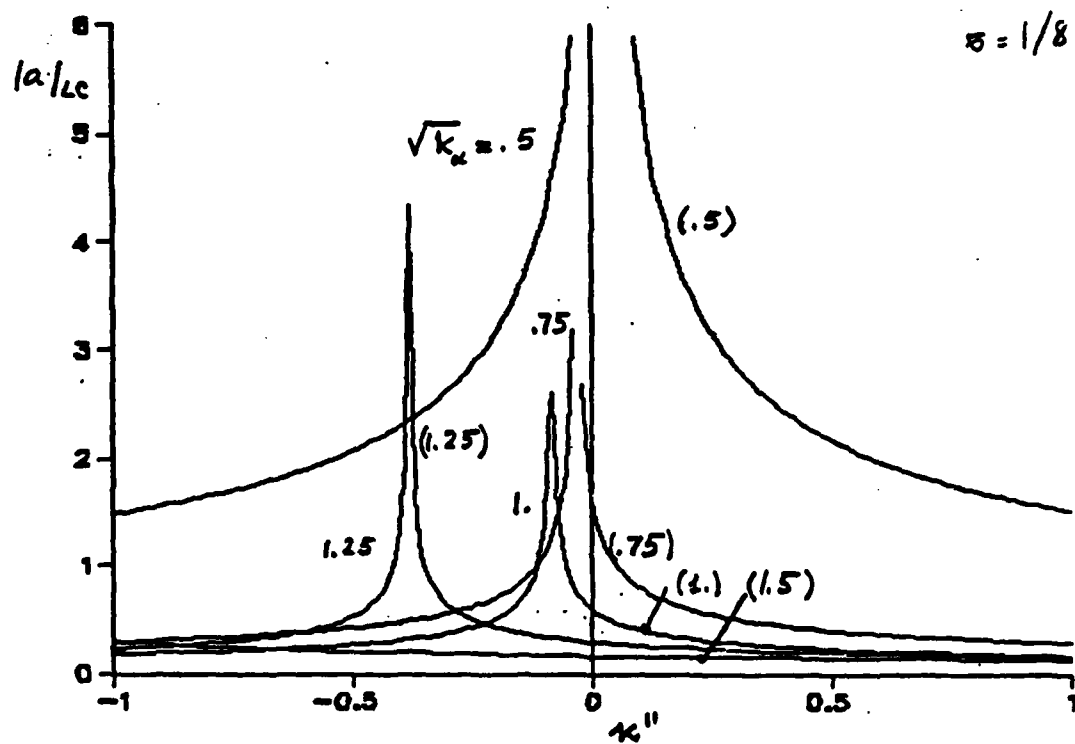
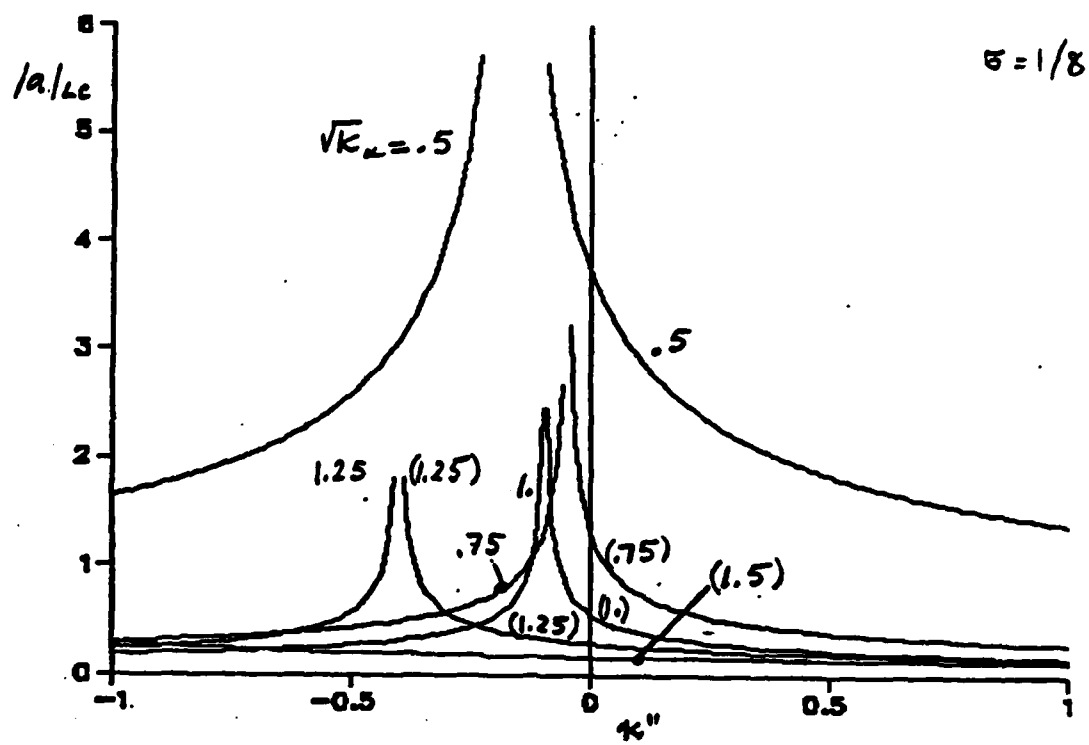


Figure 22

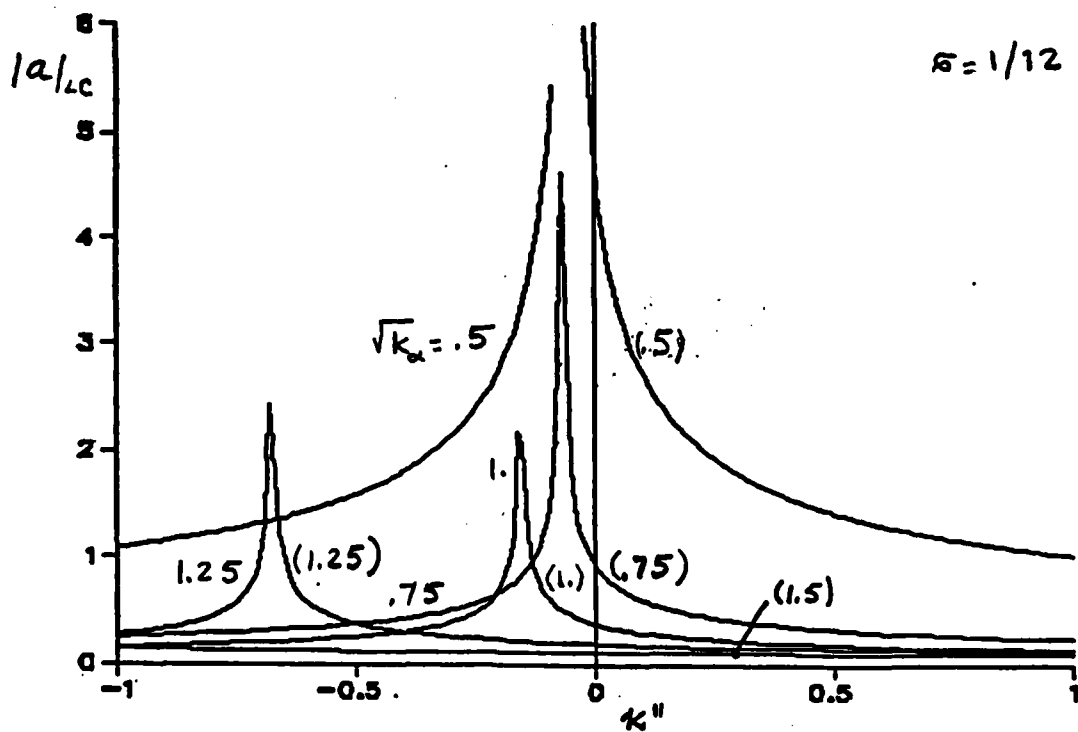
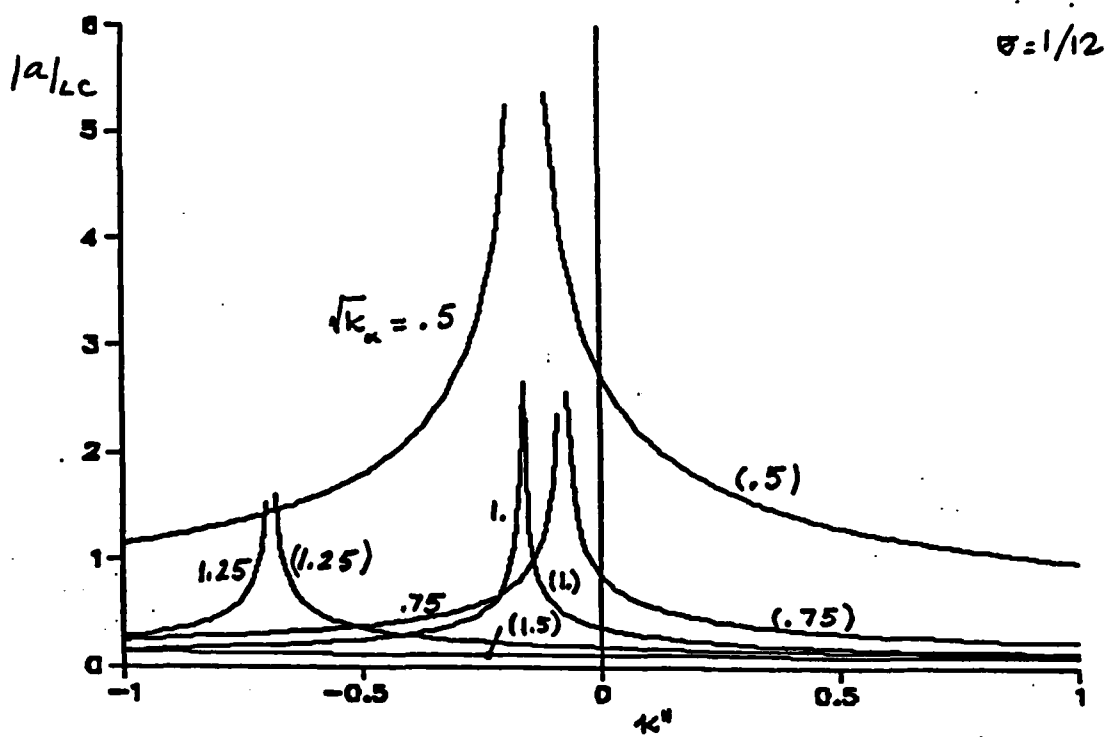


Figure 23

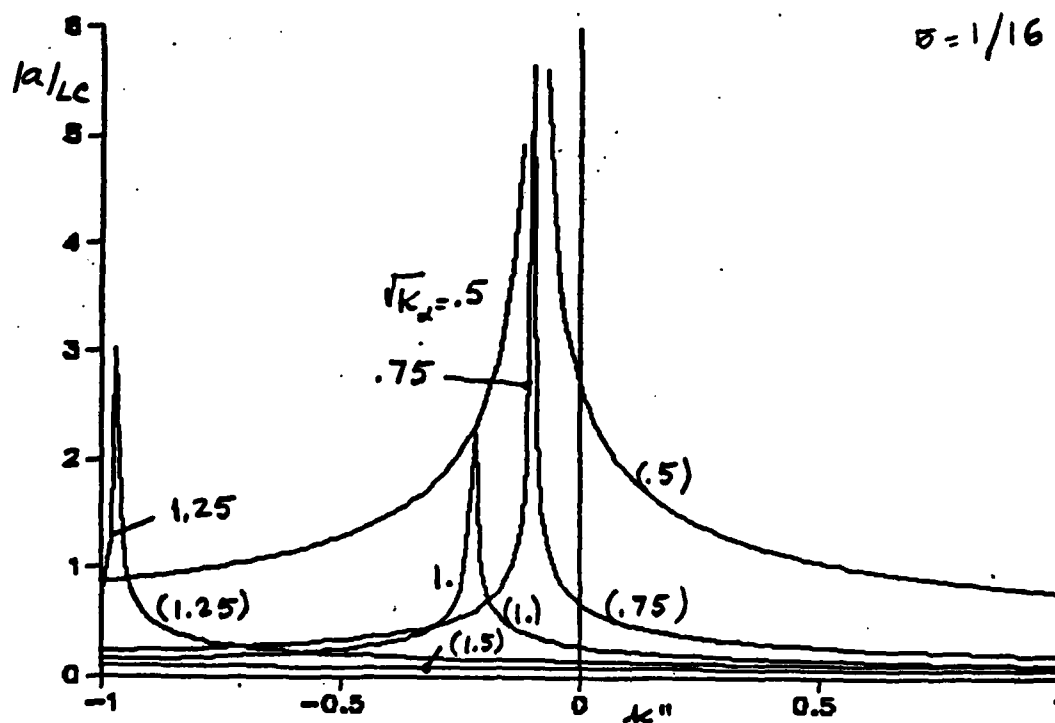
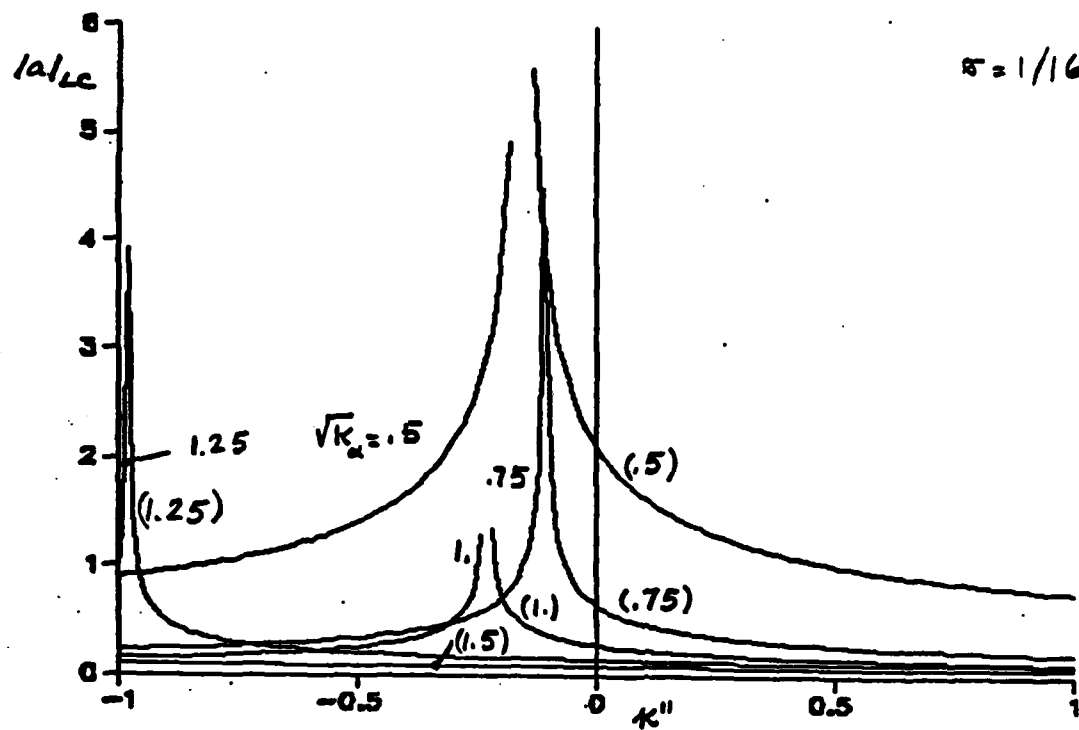


Figure 24

## SECTION VI

### CONCLUDING REMARKS

#### 6.1 Summary

A new tool for the aeroelastician, flutter taming, has been introduced. The objective of flutter taming is to insure that flutter is benign rather than destructive. This objective is accomplished by modifying or adding nonlinear terms to the governing equations, or by augmenting the system by adding state variables with a nonlinear feedback. The general theory (based on an asymptotic solution obtained by using the multiple-time-scaling method) is applied to the case of a supersonic airfoil. Taming by changing the nonlinear elastic terms and nonlinear damping terms as well as by adding a simple feedback system (a control surface with input proportional to the cube of the vertical displacement of the center of mass) have been analyzed. Analytical expressions for the solution have been derived and used to obtain numerical results. In many of the specific examples considered the results indicate that it is possible to achieve stabilization of the unstable limit cycle (which means that destructive flutter is replaced with benign flutter). Also, a dramatic reduction in the limit cycle amplitude is obtained with relatively small values for the multiplicative constant available in each case. Further increases of these constant, however, yield only marginal additional gains. As  $\delta$  decreases, however, stabilization is not achieved (within reasonable values of the multiple constants) for the higher values of  $K_2$ . The most promising results are obtained using active control.

#### 6.2 Recommendations

In view of the positive results obtained so far and in view of the great potential for practical applications, it is recommended that this work be continued by including the influence of the other parameters (such as  $K_1, \delta$  and  $\xi_0$ , see Eq. 124). It is also recommended that the work be continued with a wind tunnel simulation to confirm experimentally the results obtained here (in particular on the feasibility of using large values for the gain in the case of taming by active control). In addition the theory should be extended to include, for instance, the possibility of simultaneous (or quasi-simultaneous) crossing of the imaginary axis by two pairs of complex conjugate roots, a case that is excluded here (see Section 2.1). Other issues that should be addressed included higher order nonlinear terms, more sophisticated aerodynamic operators, three dimensional models, and adaptive control systems.

# APPENDIX A

## LAGRANGIAN EQUATIONS OF MOTION FOR AN AEROELASTIC SYSTEM

Let the displacement  $\vec{\delta}$  of a point of an aircraft be represented in terms of prescribed modes,  $\vec{M}_i$ , (such as the natural modes of vibrations or finite-element shape functions)

$$\vec{\delta}(\xi^*, t) = \sum_{i=1}^N q_i(t) \vec{M}_i(\xi^*) \quad (\text{A.1})$$

where  $t$  indicates time,  $\xi^*$  indicates a set of material (or convected) coordinates. The amplitudes,  $q_i$ , are called generalized Lagrangian coordinates of the system and satisfy the Lagrangian equations of motion

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \quad (\text{A.2})$$

where  $T$  is the kinetic energy and  $U = U(q_i)$  is the elastic energy. In addition, the generalized forces  $Q_i = Q_i^{(A)} + Q_i^{(D)}$  include the generalized aerodynamic forces

$$Q_i^{(A)} = -\lambda \oint c_p \vec{n} \cdot \vec{M}_i d\sigma \quad (\text{A.3})$$

(where  $\lambda = \frac{1}{2} \rho_\infty U_\infty^2$  is the dynamic pressure and  $c_p$  is the aerodynamic pressure coefficients) as well as the effect of structural damping,  $Q_i^{(D)}$ .

Note that

$$T = \frac{1}{2} \iiint \rho v^2 dV = \frac{1}{2} \dot{\underline{q}}^T \underline{M} \dot{\underline{q}} \quad (\text{A.4})$$

where

$$\underline{q} = \{q_i\} \quad (\text{A.5})$$

whereas

$$\underline{M} = \left[ \iiint_V \rho \vec{M}_i \cdot \vec{M}_j dV \right] = \left[ \iiint_V \rho \vec{M}_i \cdot \vec{M}_j |J| d\xi^1 d\xi^2 d\xi^3 \right] \quad (\text{A.6})$$

is called the mass matrix. Note that  $|J|$  is time independent (because of conservation of mass); therefore, the generalized masses are independent of the deformation. Therefore, Eq. A.1 ensures that  $\partial T / \partial \dot{q}_i = 0$ .

In addition, assuming that  $U$  is an analytic function of  $q_i$  and that  $q_i = 0$  corresponds to an equilibrium position (so that  $\partial U / \partial q_i = 0$  ( $i=1, \dots, N$ ) for  $q_j = 0$ ,  $j=1, \dots, N$ ) yields

$$\left\{ \frac{\partial U}{\partial q_i} \right\} = \underline{K} \underline{q} + \underline{q}^{(s)} \quad (A.7)$$

where

$$\underline{K} = \left[ \frac{\partial^2 U}{\partial q_i \partial q_j} \right]_{\underline{q}=0} \quad (A.8)$$

is called the stiffness matrix and  $\underline{q}^{(s)}$  is the vector of the nonlinear terms, which (because of the analytic nature of  $U$ ) is of the type

$$\underline{q}^{(s)} = \{ q_h^{(s)} \} = \left\{ \sum_{i=1}^N \sum_{j=1}^N b_{hij}^{(s)} q_i q_j + \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N c_{hijk}^{(s)} q_i q_j q_k + \dots \right\} \quad (A.9)$$

For simplicity, it is assumed that the aerodynamic forces are analytic functions of  $q_i$  and  $\dot{q}_i$  only, so that

$$\underline{Q}^{(A)} = \lambda (\underline{E}_0 \underline{q} + \underline{E}_1 \dot{\underline{q}}) + \underline{q}^{(A)}(\underline{x}, \lambda) \quad (A.10)$$

where  $\underline{q}^{(A)}$  is the vector of the nonlinear aerodynamic terms of the type

$$\underline{q}^{(A)} = \{ q_h^{(A)} \} = \left\{ \sum_{i=1}^{2N} \sum_{j=1}^{2N} b_{hij}^{(A)} x_i x_j + \sum_{i=1}^{2N} \sum_{j=1}^{2N} \sum_{k=1}^{2N} c_{hijk}^{(A)} x_i x_j x_k + \dots \right\} \quad (A.11)$$

where

$$(A.12)$$

$$\underline{x} = \begin{Bmatrix} \underline{q} \\ \underline{\dot{q}} \end{Bmatrix} \quad (\text{A. 12})$$

Finally, the structural-damping forces are approximated as

$$\underline{Q}^{(D)}(\underline{x}) = \underline{R} \underline{\dot{q}} + \underline{g}^{(D)}(\underline{x}) \quad (\text{A. 13})$$

where

$$\underline{g}^{(D)} = \{g_A^{(D)}\} = \left\{ \sum_{i=1}^{2N} \sum_{j=1}^{2N} b_{hij}^{(D)} x_i x_j + \sum_{i=1}^{2N} \sum_{j=1}^{2N} \sum_{k=1}^{2N} c_{hijk}^{(D)} x_i x_j x_k + \dots \right\} \quad (\text{A. 14})$$

Combining the above equations one obtains

$$\begin{aligned} \underline{M} \ddot{\underline{q}} + \underline{R} \underline{\dot{q}} + \underline{K} \underline{q} + \lambda (\underline{E}_0 \underline{q} + \underline{E}_1 \underline{\dot{q}}) \\ = \underline{g}(\underline{q}, \underline{\dot{q}}) \end{aligned} \quad (\text{A. 15})$$

where

$$\begin{aligned} \underline{g} = \{g_k\} = \left\{ \sum_{i=1}^{2N} \sum_{j=1}^{2N} b_{hij}(\lambda) x_i x_j \right. \\ \left. + \sum_{i=1}^{2N} \sum_{j=1}^{2N} \sum_{k=1}^{2N} c_{hijk}(\lambda) x_i x_j x_k + \dots \right\} \end{aligned} \quad (\text{A. 16})$$

with

$$\begin{aligned} b_{hij}(\lambda) &= b_{hij}^{(S)} + b_{hij}^{(A)}(\lambda) + b_{hij}^{(D)} \\ c_{hijk}(\lambda) &= c_{hijk}^{(S)} + c_{hijk}^{(A)}(\lambda) + c_{hijk}^{(D)} \end{aligned} \quad (\text{A. 17})$$

(where  $b_{hij}^{(S)}$  and  $c_{hijk}^{(S)}$  are understood to be equal to zero for  $i$ , or  $j$ , or  $k$  greater than  $N$ )

Equation A. 15 may be recast in a form which is convenient for the analysis of Section II. Setting

$$\underline{\dot{q}} = \underline{v} \quad (\text{A. 18})$$

Eq. A. 15 may be written as

$$\underline{M} \underline{\dot{v}} + \underline{R} \underline{v} + \underline{K} \underline{q} + \lambda \underline{E}_0 \underline{q} + \lambda \underline{E}_1 \underline{v} = \underline{g} \quad (\text{A. 19})$$



The above two equations may be combined to yield

$$\underline{J} \dot{\underline{x}} = \underline{A}(\lambda) \underline{x} + \underline{f} \quad (\text{A. 20})$$

where

$$\begin{aligned} \underline{x} &= \begin{Bmatrix} \underline{q} \\ \underline{\dot{q}} \end{Bmatrix} \\ \underline{J} &= \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{M} \end{bmatrix} \\ \underline{A}(\lambda) &= \begin{bmatrix} \underline{0} & \underline{I} \\ -\underline{K} & -\underline{R} \end{bmatrix} + \lambda \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{E}_0 & \underline{E}_1 \end{bmatrix} \\ \underline{f} &= \begin{Bmatrix} \underline{0} \\ \underline{q} \end{Bmatrix} \end{aligned} \quad (\text{A. 21})$$

Note that if the modes,  $\bar{M}_i$ , coincide with the natural modes of vibration, then

$$\begin{aligned} \underline{M} &= [\bar{m}_i] \\ \underline{K} &= [\bar{m}_i \omega_i^2] \end{aligned} \quad (\text{A. 22})$$

where  $\bar{m}_i$  and  $\omega_i$  are called the generalized masses and the natural frequencies respectively. The modes may be normalized so that  $\bar{m}_i = 1$ . In this case,  $\underline{J} = \underline{I}$  and Eq. A.20 is equal to Eq. 1. In general, premultiplication of Eq. A.20 by  $\underline{J}^{-1}$  yields an equation formally equal to Eq. 1.

If active control is included, the system is called aeroservoelastic. The control equations are usually cast in the form of Eq. A.20 so that the combined system may be easily recast in an equation formally equal to Eq. 1.

## APPENDIX B

### AIRFOIL FLUTTER

#### B.1 Formulation of Problem

Consider the airfoil shown in Figure B.1. The center of mass,  $C$ , is allowed to move only in the vertical direction: hence the motion of the airfoil is limited to two degrees of freedom: the vertical displacement,  $b\zeta$ , of the center of mass  $C$  (where  $b$  is the semichord of the airfoil), and the rotation around the center of mass  $C$ . Let  $b\xi$  and  $b\eta$  be the Cartesian coordinates in a frame of reference fixed with the airfoil (i.e.,  $\xi$  and  $\eta$  are convected coordinates) such that origin is in  $C$  (center of mass) and  $\xi$ -axis goes through the point  $E$  (point of attachment of the springs, i.e., elastic axis).

The location of a point  $(\xi, \eta)$  in an absolute frame of reference  $(x, y)$  is given by

$$x = b(\xi \cos \alpha + \eta \sin \alpha)$$

$$y = b(-\xi \sin \alpha + \eta \cos \alpha + \zeta)$$

(B.1)

The motion of the airfoil is governed by the Lagrange equation of motion

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\xi}} \right) - \frac{\partial T}{\partial \xi} + \frac{\partial U}{\partial \xi} = Q_{\xi}$$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}} \right) - \frac{\partial T}{\partial \alpha} + \frac{\partial U}{\partial \alpha} = Q_{\alpha} \quad (B.2)$$

where  $\cdot$  indicates differentiation with respect to a nondimensional time

$$t = a_{\infty} \hat{t} / b$$

(B.3)

(where  $a_{\infty}$  is the speed of sound of the undisturbed fluid) whereas  $T$  and  $U$  are the kinetic and potential (elastic) energy respectively, and  $Q_{\xi}$  and  $Q_{\alpha}$  are such that the virtual work  $\delta W$  is given as

$$\delta W = Q_{\xi} \delta \xi + Q_{\alpha} \delta \alpha \quad (B.4)$$

In order to obtain the expression for the kinetic energy, note that

$$u = a_{\infty} \dot{x} / b = a_{\infty} (-\xi \sin \alpha + \eta \cos \alpha) \dot{\alpha}$$

$$v = a_{\infty} \dot{y} / b = a_{\infty} (-\xi \cos \alpha - \eta \sin \alpha) \dot{\alpha} + a_{\infty} \dot{\zeta}$$

(B.5)

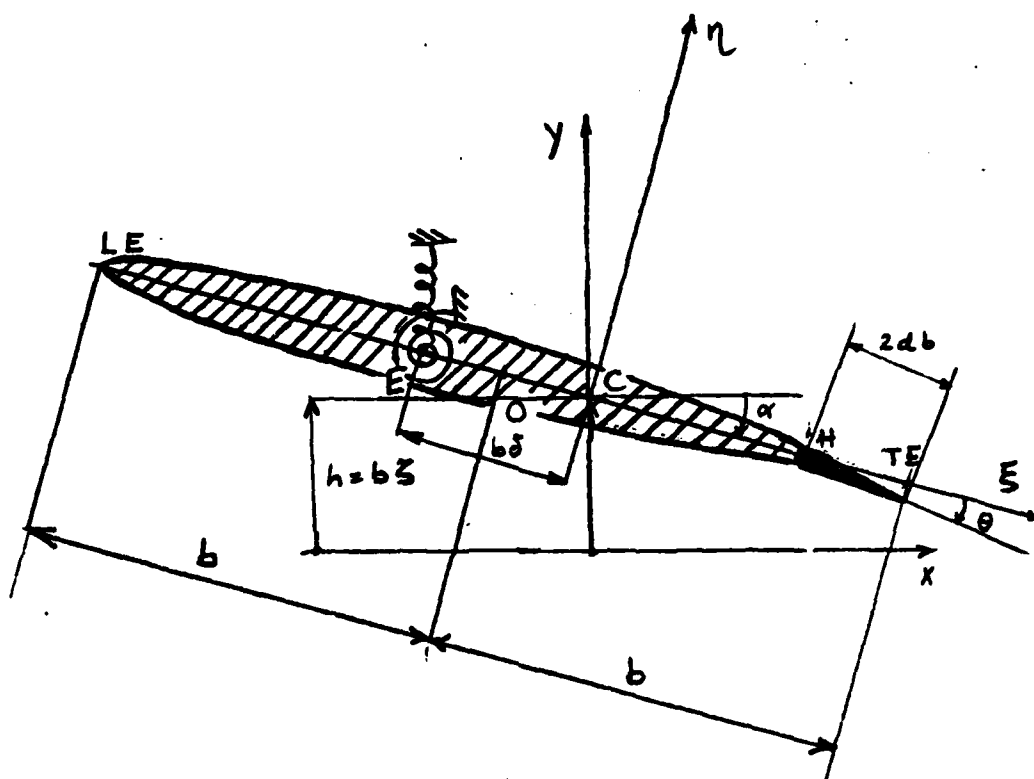


Figure B-1

Hence

$$\begin{aligned}
 T &= \frac{1}{2} \iint \rho (u^2 + v^2) dA \\
 &= \frac{1}{2} a_\omega^2 b^2 \iint \rho \left\{ [(-\xi \sin \alpha + \eta \cos \alpha) \dot{\alpha}]^2 \right. \\
 &\quad \left. + [(-\xi \cos \alpha - \eta \sin \alpha) \dot{\alpha} + \dot{\xi}]^2 \right\} d\xi d\eta \\
 &= \frac{1}{2} a_\omega^2 b^2 \left[ \dot{\alpha}^2 \iint \rho (\xi^2 + \eta^2) d\xi d\eta + 2 \dot{\alpha} \dot{\xi} \iint \rho (-\xi \cos \alpha - \eta \sin \alpha) d\xi d\eta \right. \\
 &\quad \left. + \dot{\xi}^2 \iint \rho d\xi d\eta \right] \quad (B.6)
 \end{aligned}$$

or noting that, by definition of center of mass, C (recall that  $\xi_c = \eta_c = 0$ ),

$$\iint \rho \xi d\xi d\eta = \iint \rho \eta d\xi d\eta \quad (B.7)$$

one obtains the desired expression for the kinetic energy

$$T = \frac{1}{2} m a_\omega^2 \dot{\xi}^2 + \frac{1}{2} \frac{I a_\omega^2}{b^2} \dot{\alpha}^2 \quad (B.8)$$

where

$$m = b^2 \iint \rho d\xi d\eta \quad (B.9)$$

is the mass of the airfoil, whereas

$$I = b^4 \iint \rho (\xi^2 + \eta^2) d\xi d\eta \quad (B.10)$$

is the moment of inertia around C.

Next consider the potential energy  $U$ . The motion is assumed to be restrained by two springs attached at point E. For the first spring the potential energy  $U$  is a function of the nondimensional vertical displacement  $\zeta_E = h_E/b$  at E and the second one, the potential energy is a function of the rotation  $\alpha$ . Hence the potential energy is given by

$$U = U_\zeta(\zeta_E) + U_\alpha(\alpha) \quad (B.11)$$

Recalling that the  $\xi$ -axis goes through E, so that  $\eta_E = 0$ , one obtains

$$\xi_E = h_E / b = b(\xi + \delta \sin \alpha) \quad (\text{B. 12})$$

with

$$\delta = -\xi_E \quad (\text{B. 13})$$

Next, note that if  $p$  is the pressure and  $\hat{s}$  the arclength along the airfoil,

$$\begin{aligned} \delta W &= \oint (-p \vec{n} \cdot \delta \vec{x}) d\hat{s} \\ &= - \oint p (n_x \delta x + n_y \delta y) d\hat{s} \end{aligned} \quad (\text{B. 14})$$

Using Eq. 1 and

$$\begin{aligned} n_x &= n_\xi \cos \alpha + n_\eta \sin \alpha \\ n_y &= -n_\xi \sin \alpha + n_\eta \cos \alpha \end{aligned} \quad (\text{B. 15})$$

(where  $n_\xi$  and  $n_\eta$  are the components of  $\vec{n}$  in the  $(\xi, \eta)$  frame of reference) one obtains (see Eq. B. 4)

$$\begin{aligned} \delta W &= -b^2 \oint p [(n_\xi \cos \alpha + n_\eta \sin \alpha)(-\xi \sin \alpha \delta \alpha + \eta \cos \alpha \delta \alpha) \\ &\quad + (-n_\xi \sin \alpha + n_\eta \cos \alpha)(-\xi \cos \alpha \delta \alpha - \eta \sin \alpha \delta \alpha + b \delta \xi)] ds \\ &= -b^2 \oint p [(\eta n_\xi - \xi n_\eta) \delta \alpha + (-n_\xi \sin \alpha + n_\eta \cos \alpha) \delta \xi] ds \\ &= Q_\xi d\xi + Q_\alpha d\alpha \end{aligned} \quad (\text{B. 16})$$

with  $ds = d\hat{s}/b$  and

$$\begin{aligned} Q_\xi &= -b^2 \oint p (-n_\xi \sin \alpha + n_\eta \cos \alpha) ds = -b^2 \oint p n_y ds \\ Q_\alpha &= -b^2 \oint p (\eta n_\xi - \xi n_\eta) ds \end{aligned} \quad (\text{B. 17})$$

Finally, combining Eqs. B. 2, B. 8, B. 11 and B. 12, one obtains

$$\begin{aligned} \ddot{\xi} + \frac{\partial V}{\partial \xi} &= e_z \\ \nu \ddot{\alpha} + \frac{\partial V}{\partial \alpha} &= e_\alpha \end{aligned} \quad (\text{B. 18})$$

with

$$V = U / m a_\infty^2$$

$$e_z = Q_z / m a_\infty^2$$

$$e_x = Q_x / m a_\infty^2$$

$$v = I / m b^2$$

(B. 19)

where  $Q_z$  and  $Q_x$  are given by Eq. B. 17 and  $U$  is defined by Eq. B. 11.

## B. 2 Aerodynamic Forces

In order to complete the formulation, we need an expression for the pressure distribution. We will assume that the air flow around the airfoil is supersonic and that the pressure is given by the piston-theory expression (Ref. 12)

$$\frac{p}{p_\infty} = \left( 1 + \frac{\gamma-1}{2} \frac{\psi}{a_\infty} \right)^{\frac{2\gamma}{\gamma-1}} \quad (\text{B. 20})$$

where  $a_\infty$  is the undisturbed speed of sound,  $\gamma = c_p/c_v$  is the specific heat ratio and  $\psi$  is the normal-wash (velocity component in the direction normal to the surface), that is

$$\begin{aligned} \psi/U_\infty &= (-U_\infty \vec{i} + d\vec{x}/d\hat{x}) \cdot \vec{n}/U_\infty = -n_x + (\dot{x} n_x + \dot{y} n_y) a_\infty / b U_\infty \\ &= -(n_\xi \cos \alpha + n_\eta \sin \alpha) + \frac{1}{M} (-\xi \sin \alpha + \eta \cos \alpha) \dot{\alpha} (n_\xi \cos \alpha + n_\eta \sin \alpha) \\ &\quad + \frac{1}{M} [(-\xi \cos \alpha - \eta \sin \alpha) \dot{\alpha} + \dot{\xi}] (-n_\xi \sin \alpha + n_\eta \cos \alpha) \\ &= -(n_\xi \cos \alpha + n_\eta \sin \alpha) + \frac{1}{M} (-n_\xi \sin \alpha + n_\eta \cos \alpha) \dot{\xi} \\ &\quad + \frac{1}{M} (n_\xi \eta - n_\eta \xi) \dot{\alpha} \end{aligned} \quad (\text{B. 21})$$

where

$$M = U_\infty / a_\infty \quad (\text{B. 22})$$

For the sake of simplicity, in the following we will assume that the airfoil has zero-thickness and zero-camber. In this case,  $\eta = 0$ . Equations B.1 and B.4 remain essentially the same (except for the fact that  $\eta = 0$ ). In particular, Eq. B.1 reduce to

$$\begin{aligned}x &= b \xi \cos \alpha \\y &= -b \xi \sin \alpha + b \zeta\end{aligned}\quad (\text{B.23})$$

Next, note that

$$\begin{aligned}n_\xi &= 0 \\n_\eta &= +1 \quad (\text{upper side}) \\&= -1 \quad (\text{lower side})\end{aligned}\quad (\text{B.24})$$

Using these equations, Eq. B.19 reduces to

$$\begin{aligned}e_s &= \frac{\epsilon}{4\gamma} \cos \alpha \int \frac{\Delta p}{p_\infty} d\xi \\e_n &= -\frac{\epsilon}{4\gamma} \int \frac{\Delta p}{p_\infty} \xi d\xi\end{aligned}\quad (\text{B.25})$$

where the pressure discontinuity is given by

$$\Delta p = p_{\text{upper}} - p_{\text{lower}} \quad (\text{B.26})$$

the mass ratio  $\epsilon$  is defined as (recall that  $a_\infty^2 = \gamma p_\infty / \rho_\infty$ )

$$\epsilon = \frac{4\gamma b^2 p_\infty}{m a_\infty^2} = \frac{4 b^2 \rho_\infty}{m} \quad (\text{B.27})$$

and the integral are understood from leading edge to trailing edge. The value of the nondimensional normalwash evaluated on the upper side of the airfoil is given by

$$\Psi = \frac{1}{U_\infty} \psi_{\text{upper}} = -\sin \alpha + \frac{1}{M} \cos \alpha \xi - \frac{1}{M} \xi \alpha \quad (\text{B.28})$$

(the opposite sign holds for the lower side of the airfoil).

The pressure may be written as

$$\frac{p}{p_\infty} = 1 + \gamma \frac{\psi}{a_\infty} + \frac{1}{4} \gamma(\gamma+1) \left(\frac{\psi}{a_\infty}\right)^2 + \frac{1}{12} \gamma(\gamma+1) \left(\frac{\psi}{a_\infty}\right)^3 + \dots \quad (\text{B.29})$$

and (noting that  $\psi$  has opposite sign on the upper and lower sides) the pressure difference is given by

$$\Delta p/p_\infty = 2\gamma (M\psi + \chi M^3\psi^3 + \dots) \quad (\text{B. 30})$$

with  $\chi = (\gamma+1)/2$ .

Combining Eqs. B. 28 and B. 30 one obtains

$$\begin{aligned} \Delta p/p_\infty = 2\gamma \left[ \left(-\alpha + \frac{1}{6}\alpha^3\right)M + \left(1 - \frac{\alpha^2}{2}\right)\dot{\xi} - \xi\dot{\alpha} \right. \\ \left. - \chi (M\alpha - \dot{\xi} + \xi\dot{\alpha})^3 + \dots \right] \end{aligned} \quad (\text{B. 31})$$

and, substituting in Eq. B. 25,

$$\begin{aligned} e_\xi &= -\sigma \left(1 - \frac{\alpha^2}{2}\right) \frac{1}{2} \int (-M\alpha + \dot{\xi} - \xi\dot{\alpha} + \frac{1}{6}M\alpha^3 - \frac{1}{2}\alpha^2\dot{\xi}) d\xi \\ &\quad + \sigma\chi \frac{1}{2} \int (M\alpha - \dot{\xi} + \xi\dot{\alpha})^3 d\xi + \dots \\ &= -\sigma \left[ -MS_0\alpha + S_0\dot{\xi} - S_1\alpha + \frac{2}{3}S_0M\alpha^3 - S_0\alpha^2\dot{\xi} + \frac{1}{2}S_1\alpha^2\dot{\alpha} \right] \\ &\quad + \sigma\chi \left[ S_0\omega^3 + 3S_1\omega^2\dot{\alpha} + 3S_2\omega\dot{\alpha}^2 + S_3\dot{\alpha}^3 \right] + \dots \\ e_\alpha &= \sigma \frac{1}{2} \int (-M\alpha + \dot{\xi} - \xi\dot{\alpha} + \frac{1}{6}M\alpha^3 - \frac{1}{2}\alpha^2\dot{\xi}) \xi d\xi \\ &\quad - \sigma\chi \frac{1}{2} \int (M\alpha - \dot{\xi} + \xi\dot{\alpha})^3 \xi d\xi + \dots \\ &= \sigma \left[ -MS_1\alpha + S_1\dot{\xi} - S_2\dot{\alpha} + \frac{1}{6}S_1M\alpha^3 - \frac{1}{2}S_1\alpha^2\dot{\xi} \right] \\ &\quad - \sigma\chi \left[ S_1\omega^3 + 3S_2\omega^2\dot{\alpha} + 3S_3\omega\dot{\alpha}^2 + S_4\dot{\alpha}^4 \right] + \dots \end{aligned} \quad (\text{B. 32})$$

with

$$\omega = M\alpha - \dot{\xi} \quad (\text{B. 33})$$

and

$$\begin{aligned} S_n &= \frac{1}{2} \int_{-1+\xi_0}^{1+\xi_0} \xi^n d\xi = 1 & (n=0) \\ &= \xi_0 & (n=1) \\ &= (1 + 3\xi_0^2)/3 & (n=2) \\ &= \xi_0 + \xi_0^3 & (n=3) \\ &= (1 + 10\xi_0^2 + 5\xi_0^4)/5 & (n=4) \end{aligned} \quad (\text{B. 34})$$

where  $\xi_0$  is the value of  $\xi$  at midchord.



Equations B.32 and B.33 may be recast as

$$\begin{Bmatrix} e_5 \\ e_\alpha \end{Bmatrix} = -\epsilon \begin{bmatrix} S_0 & -S_1 \\ -S_1 & S_2 \end{bmatrix} \begin{Bmatrix} \dot{\xi} \\ \dot{\alpha} \end{Bmatrix} - \lambda \begin{bmatrix} 0 & -S_0 \\ 0 & S_1 \end{bmatrix} \begin{Bmatrix} \xi \\ \alpha \end{Bmatrix} + \begin{Bmatrix} g_5^{(A)} \\ g_\alpha^{(A)} \end{Bmatrix} + \dots \quad (\text{B.35})$$

where

$$\lambda = M\epsilon = \frac{U_\infty}{a_\infty} \frac{4\rho_\infty b^2}{m} \quad (\text{B.36})$$

whereas

$$\begin{aligned} g_5^{(A)} &= \epsilon \left( -\frac{2}{3} S_0 M \alpha^3 + S_0 \alpha^2 \dot{\xi} - \frac{1}{2} S_1 \alpha^2 \dot{\alpha} \right) \\ &\quad + \epsilon \chi (S_0 w^3 + 3S_1 w^2 \dot{\alpha} + 3S_2 w \dot{\alpha}^2 + S_3 \dot{\alpha}^3) \\ g_\alpha^{(A)} &= \epsilon \left( \frac{1}{6} S_1 M \alpha^3 - \frac{1}{2} S_1 \alpha^2 \dot{\xi} \right) \\ &\quad - \epsilon \chi (S_1 w^3 + 3S_2 w^2 \dot{\alpha} + 3S_3 w \dot{\alpha}^2 + S_4 \dot{\alpha}^3) \end{aligned} \quad (\text{B.37})$$

It may be noted that in the formulation used by Smith and Morino (Ref. 15)  $\psi$  represents the downwash (velocity in the direction of the  $y$ -axis) instead of the normalwash (velocity in the direction of the  $\eta$ -axis); hence the expression for  $\psi$  consistent with the formulation of Ref. 15 is

$$\psi^{(SM)} = -\tan \alpha + \frac{\dot{\xi}}{M} - \frac{1}{M} \xi \dot{\alpha} \cos \alpha \quad (\text{B.38})$$

(see Eq. C6 of Ref. 15)

Using this expression one obtains that  $e_5$  and  $e_\alpha$  are still given by Eq. B.35 with

$$\begin{aligned} g_5^{(A)} &= \epsilon \left( \frac{1}{6} S_0 M \alpha^3 + \frac{1}{2} S_0 \alpha^2 \dot{\xi} - S_1 \alpha^2 \dot{\alpha} \right) \\ &\quad + \epsilon \chi (S_0 w^3 + 3S_1 w^2 \dot{\alpha} + 3S_2 w \dot{\alpha}^2 + S_3 \dot{\alpha}^3) \end{aligned} \quad (\text{B.39})$$

and

$$\begin{aligned} g_\alpha^{(A)} &= \epsilon \left( -\frac{1}{3} S_1 M \alpha^3 + \frac{1}{2} S_2 \alpha^2 \dot{\alpha} \right) \\ &\quad + \epsilon \chi (S_1 w^3 + 3S_2 w^2 \dot{\alpha} + 3S_3 w \dot{\alpha}^2 + S_4 \dot{\alpha}^3) \end{aligned} \quad (\text{B.40})$$

respectively (in agreement with Eqs C12c and C12d of Ref. 15)

### B.3 Elastic Forces

For the sake of simplicity the problem is limited to the case in which the potential energies  $U_5$  and  $U_\alpha$  are even

functions of  $\zeta_E$  and  $\alpha$  respectively (this implies for instance that the axial spring acts symmetrically in compression and tension), and are given in the forms

$$U_{\zeta} = 1/2 K_{\zeta} h_E^2 + 1/4 K'_{\zeta} h_E^4 + \dots$$

$$U_{\alpha} = 1/2 K_{\alpha} \alpha^2 + 1/4 K'_{\alpha} \alpha^4 + \dots \quad (B.41)$$

Using Eqs. B.12 and B.19 one obtains

$$\partial V_{\zeta} / \partial \zeta_E = k_{\zeta} \zeta_E + k'_{\zeta} \zeta_E^3 + \dots \quad (B.42)$$

and

$$\partial V_{\alpha} / \partial \alpha = k_{\alpha} \alpha + k'_{\alpha} \alpha^3 \quad (B.43)$$

where

$$\begin{aligned} k_{\zeta} &= K_{\zeta} b^2 / m a_w^2 & k_{\alpha} &= K_{\alpha} / m a_w^2 \\ k'_{\zeta} &= K'_{\zeta} b^4 / m a_w^2 & k'_{\alpha} &= K'_{\alpha} / m a_w^2 \end{aligned} \quad (B.44)$$

(note that  $k_{\zeta}$ ,  $k_{\alpha}$ ,  $k'_{\zeta}$ , and  $k'_{\alpha}$  coincide respectively with  $\Omega_{\zeta}^2, \Omega_{\alpha}^2, \kappa_{\zeta}, \kappa_{\alpha}$  of Ref. 15)

Using Eqs. B.12, B.42 and B.43 yields

$$\partial V / \partial \zeta = \partial V_{\zeta} / \partial \zeta_E = k_{\zeta} (\zeta + \delta \alpha - 1/6 \delta \alpha^3 + \dots) + k'_{\zeta} (\zeta + \delta \alpha + \dots)^3 + \dots \quad (B.45)$$

and

$$\begin{aligned} \partial V / \partial \alpha &= (\partial V_{\zeta} / \partial \zeta_E) \delta \cos \alpha + \partial V_{\alpha} / \partial \alpha \\ &+ [(k_{\zeta} \zeta + \delta \alpha - 1/6 \delta \alpha^3 + \dots) + k'_{\zeta} (\zeta + \delta \alpha + \dots)^3] \delta (1 - \frac{\alpha^2}{2} + \dots) + k_{\alpha} \alpha + k'_{\alpha} \alpha^3 + \dots \end{aligned} \quad (B.46)$$

These equations may be rewritten as

$$\begin{Bmatrix} \partial V / \partial \zeta \\ \partial V / \partial \alpha \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \zeta \\ \alpha \end{Bmatrix} - \begin{Bmatrix} g_{\zeta}^{(s)} \\ g_{\alpha}^{(s)} \end{Bmatrix} + \dots \quad (B.47)$$

where

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} k_{\zeta} & k_{\zeta} \delta \\ k_{\zeta} \delta & k_{\zeta} \delta^2 + k_{\alpha} \end{bmatrix} \quad (B.48)$$

and

$$g_{\zeta}^{(s)} = 1/6 k_{\zeta} \delta \alpha^3 - k'_{\zeta} (\zeta + \delta \alpha)^3 \quad (B.49)$$

$$g_{\alpha}^{(s)} = 1/2 k_{\zeta} \delta \zeta \alpha^2 + (2/3 k_{\zeta} \delta^2 - k'_{\alpha}) \alpha^3 - k'_{\zeta} \delta (\zeta + \delta \alpha)^3 \quad (B.50)$$

in agreement with Ref. 15 (Eqs. C12a and C12b).

#### B.4 Governing Equations

Combining Eqs. B.1B, B.35, and B.47 one obtains

$$\underline{M}\ddot{\underline{y}} + \underline{R}\dot{\underline{y}} + (\underline{K} + \lambda \underline{E})\underline{y} = \underline{g} \quad (\text{B.51})$$

where

$$\underline{y} = \begin{Bmatrix} \xi \\ \alpha \end{Bmatrix} \quad (\text{B.52})$$

$$\underline{M} = \begin{bmatrix} 1 & 0 \\ 0 & \nu \end{bmatrix} \quad (\text{B.53})$$

$$\underline{R} = \begin{bmatrix} \epsilon S_0 & -\epsilon S_1 \\ -\epsilon S_1 & \epsilon S_2 \end{bmatrix} \quad (\text{B.54})$$

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \quad (\text{B.55})$$

$$\underline{E} = \begin{bmatrix} 0 & -S_0 \\ 0 & S_1 \end{bmatrix} \quad (\text{B.56})$$

and

$$\underline{g} = \begin{Bmatrix} g_\xi \\ g_\alpha \end{Bmatrix} = \begin{Bmatrix} g_\xi^{(A)} + g_\xi^{(S)} \\ g_\alpha^{(A)} + g_\alpha^{(S)} \end{Bmatrix} \quad (\text{B.57})$$

#### B.5 Modified System

In order to achieve flutter-taming, the system may be modified for instance by changing the design to obtain different values of the nonlinear stiffness coefficients  $k'_\xi$  and  $k'_\alpha$ . If the structure is already built (such as in the case of existing airplane) one could add a nonlinear damper, causing for instance a moment proportional to  $\dot{\alpha}^3$ , i. e., modifying the expression for  $g_\alpha$  (Eq. B.57) as

$$g_\alpha = g_\alpha^{(A)} + g_\alpha^{(S)} + g_\alpha^{(D)} \quad (\text{B.58})$$

with

$$g_\alpha^{(D)} = -\kappa' \dot{\alpha}^3 \quad (\text{B.59})$$

A more ambitious approach (an active control modification) may be obtained by using a control surface. Indicating with the deflection of the control surface, the normalwash is given by (see Eq. B.2B)

$$\begin{aligned}\Psi &= -\sin\alpha + (\cos\alpha \cdot \dot{\xi} - \xi \dot{\alpha})/M & (\xi_H \leq \xi < \xi_H) & \quad (B.60) \\ &= -\sin(\alpha + \theta) + [\cos(\alpha + \theta) - \xi \dot{\alpha} - (\xi - \xi_H)\dot{\theta}]/M & (\xi_H < \xi \leq \xi_{TE}) & \quad (B.60)\end{aligned}$$

where  $\xi_H$  is the location of the hinge line. For the analysis presented here (in which  $\theta$  is of the order of  $\alpha^3$  or  $\alpha^2$ ) the above expression may be approximated by

$$\begin{aligned}\Psi &= -\alpha + 1/6 \alpha^3 + (\dot{\xi} - \xi \dot{\alpha} - 1/2 \alpha^2 \dot{\xi})/M \\ &\quad - H(\xi - \xi_H) \theta - R(\xi - \xi_H) \dot{\theta} + \dots & (B.61)\end{aligned}$$

where  $H(u)$  is the Heaviside function and  $R(u) = uH(u)$  is the ramp function. The last two terms yield an additional pressure distribution,  $p_\theta$  (due to the control surface deflection  $\theta$ ), equal to

$$p_\theta = -2\gamma [M H(\xi - \xi_H) \theta + R(\xi - \xi_H) \dot{\theta}] \quad (B.62)$$

This yields an increase in  $q_\xi$  and  $q_\alpha$  equal to

$$\begin{aligned}\Delta q_\xi &= \epsilon (M S_\xi^{(n)} \theta + S_\xi^{(n)} \dot{\theta}) \\ \Delta q_\alpha &= -\epsilon (M S_\alpha^{(n)} \theta + S_\alpha^{(n)} \dot{\theta}) & (B.63)\end{aligned}$$

where

$$S_\xi^{(n)} = \frac{1}{2} \int_{\xi_H}^{\xi_{TE}} (\xi - \xi_H)^n d\xi = \frac{1}{2} \int_0^{2d} u^n du = d \quad (n=0) \\ = d^2 \quad (n=1) \quad (B.65)$$

$$\begin{aligned}\text{and } S_\alpha^{(n)} &= \frac{1}{2} \int_{\xi_H}^{\xi_{TE}} \xi (\xi - \xi_H)^n d\xi = \frac{1}{2} \int_0^{2d} (\xi_H + u) u^n du = d(\xi_H + d) \quad (n=0) \\ &= d^2 \left( \xi_H + \frac{2}{3} d \right) \quad (n=1) & (B.66)\end{aligned}$$

where  $d = (\xi_{TE} - \xi_H)/2$  is the ratio between the chord of the control surface and the chord of the airfoil.

The dynamics of the control surface is assumed to be to irreversible (i.e., unaffected by the aerodynamic forces) and governed by

$$J \ddot{\theta} + r \dot{\theta} + k \theta = g_0^{(F)} \quad (B.67)$$

where the feedback term  $g_0^{(F)}$  is proportional for instance to the vertical displacement of the center of mass:

$$g_0^{(F)} = 10^4 \zeta^3 \quad (\text{B. 68})$$

The resulting equations are

$$\underline{M} \ddot{\underline{z}} + \underline{R} \dot{\underline{z}} + (\underline{K} + \lambda \underline{E}) \underline{z} = \underline{g} \quad (\text{B. 69})$$

where

$$\underline{M} = \begin{Bmatrix} \zeta \\ \lambda \\ 0 \end{Bmatrix} \quad (\text{B. 70})$$

$$\underline{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & J \end{bmatrix} \quad (\text{B. 71})$$

$$\underline{R} = \begin{bmatrix} cS_0 & -cS_1 & -cS_2^{(1)} \\ -cS_1 & cS_2 & cS_2^{(1)} \\ 0 & 0 & r \end{bmatrix} \quad (\text{B. 72})$$

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{12} & k_{22} & 0 \\ 0 & 0 & k \end{bmatrix} \quad (\text{B. 73})$$

$$\underline{E} = \begin{bmatrix} 0 & -S_0 & -S_2^{(2)} \\ 0 & S_1 & S_2^{(2)} \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B. 74})$$

$$\underline{g} = \begin{Bmatrix} g_5^{(A)} \\ g_a^{(A)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} g_5^{(s)} \\ g_a^{(s)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ g_a^{(D)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ g_0^{(F)} \end{Bmatrix} \quad (\text{B. 75})$$

with  $g_a^{(D)}$  given by Eq. B. 62 and  $g_0^{(F)}$  given by Eq. B. 70

It should be emphasized that the three modifications introduced in this section (a. modify  $k'_2$  and/or  $k'_a$ , b. add a damper producing a moment proportional to  $\dot{\alpha}^3$ , and c. nonlinear feedback used in the dynamic equation for deflection of the control surface) are only examples. Many other modifications, equally valid, can be devised (such as a damper creating a vertical force proportional to  $\dot{\alpha}^3$ ). However these examples should be sufficient to give a good idea of the flexibility available in implementing the idea of flutter taming.

## REFERENCES

1. Bisplinghoff, R.L., Ashley, H., Hoffman, R.L., Aeroelasticity, Addison-Wesley Publishing Company, Inc., 1955.
2. Garrick, I.E., "Aeroelasticity-Frontiers and Beyond, 13th von Karman Lecture", Journal of Aircraft, Vol. 13, No. 9, Sept. 1976, pp. 641-657.
3. Edwards, J.W., "Unsteady Aerodynamics Modeling and Active Aeroelastic Control", Stanford University Center for System Research, SUDAAR 504, Feb. 1977.
4. Librescu, L., "Aeroelastic Stability of Orthotropic Heterogeneous Thin Panels in the Vicinity of the Flutter Critical Boundary, Part I", J. Mechanics IV, 1965, pp. 51-76.
5. Kobayashi, S., "Flutter of Simply Supported Rectangular Panels in a Supersonic Flow-Two-Dimensional Panel Flutter, I. Simply Supported Panels, II. Clamped Panels", Trans. Japan Soc. Aeronaut. Sci., Vol. 5, pp. 79-118.
6. Dowell, E.H., "Nonlinear Oscillation of a Fluttering Plate", AIAA J., Vol. 4, 1966, pp. 1267-1275.
7. Dowell, E.H., "Nonlinear Oscillation of a Fluttering Plate II", AIAA J., Vol. 5, 1967, pp. 1856-1862.
8. Dowell, E.H., "Panel Flutter: A Review of the Aeroelastic Stability of Plates and Shells", AIAA J., Vol. 8, No. 3, March 1970, pp. 385-399.
9. Dowell, E.H., Aeroelasticity of Plates and Shells, Noordhoff International Publishing, Leyden, 1975.
10. Dowell, E.H., et al., A Modern Course in Aeroelasticity, Sijthoff and Noordhoff, 1978.
11. Morino, L., "A Perturbation Method for Treating Nonlinear Panel Flutter Problems", AIAA J., Vol. 7, 1969, pp. 405-411.
12. Sandri, G., "The Foundations of Nonequilibrium Statistical Mechanics, I", Annals of Physics, Vol. 24, Oct. 1963, pp. 332-379.

13. Cole, J., and Kevorkian, J., "Uniformly Valid Asymptotic Approximation for Certain Nonlinear Differential Equations", Nonlinear Differential Equations in Nonlinear Mechanics, Academic Press, New York, Vol. 9, 1963, pp. 113-120.
14. Nayfeh, J.A., "A Perturbation Method for Treating Nonlinear Oscillation Problems", Journal of Mathematics and Physics, Vol. XLIV, 1965, pp. 368-374.
15. Smith, L.L., and Morino, L., "Stability Analysis of Nonlinear Differential Autonomous Systems with Applications to Flutter", AIAA J., Vol. 14, 1976, pp. 333-341.
16. Morino, L., and Kuo, C.C., "Detailed Extension of Perturbation Method for Panel Flutter", ASRL TR-164-2, March 1971, Aeroelastic and Structures Research Lab., Dept. of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Mass.
17. Davis, W.H., Jr., "Nonlinear Flutter of Panels and Wings by Lie Transformation Perturbation Method", Boston University, Dept. of Aerospace Engineering, TR-73-04, 1973.
18. Ashley, H., and Zartarian, G., "Piston Theory - A New Aerodynamic Tool for the Aeroelastician," Journal of the Aeronautical Sciences, Vol. 23, No. 12, Dec. 1956, pp. 1109-1118.

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